Positive and Projective Geometry

by

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Abstract

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This dissertation contributes to the field of positive geometry, which studies semialgebraic subsets of real algebraic varieties defined by natural positivity conditions. This is an emerging field of mathematics inspired both by computations in particle physics, as well as examples in combinatorics and representation theory. The work is divided into two parts.

The first part focuses on the amplituhedron, a semialgebraic subset of a real Grassmannian introduced by Arkani-Hamed and Trnka to encode the singularities of certain scattering amplitudes. After reviewing the basic properties of the amplituhedron, we investigate two related families of objects. First, we introduce Grasstopes, which generalize amplituhedra by removing positivity assumptions on the defining data; in the simplest case, we give a complete description of their geometry and show that they can exhibit nontrivial topology, including non-orientability. Second, we study the convex hull of the amplituhedron in its ambient Plücker space. This defines a polytope, which we call the *exterior cyclic polytope*. We study its facet structure and use it to define a *dual amplituhedron* in certain low-dimensional cases.

An important role in the theory of positive geometry is played by *positive geometries*. These are mathematical objects which capture features of the positive Grassmannian and (conjecturally) the amplituhedron. The final thing we do in this first part is provide a new family of positive geometries called *wondertopes*. These generalize the moduli space of points on the projective line. They also give an example of a positive geometry without a log canonical resolution, showing that positive geometries do not possess such a resolution in general.

The second part of this thesis connects positive geometry with classical projective geometry, of the sort which was popular in the late nineteenth and early twentieth century. In particular, we generalize the classical *Chow form* of Chow and van der Waerden, which associates to any projective variety a hypersurface in a Grassmannian. This is an important construction in algebraic geometry, which was historically invented to create a moduli space of algebraic varieties, and has more recently found applications in computer vision and metric algebraic

geometry. Our construction, the *Chow–Lam form*, associates to a subvariety of an arbitrary Grassmannian a hypersurface in another Grassmannian. We establish its basic properties and describe how it behaves under common operations such as intersection and join. Moreover, we show that certain polynomials appearing in the theory of amplituhedra arise as Chow–Lam forms of positroid varieties. We also explore applications to computational algebraic geometry, including problems in computer vision and configuration spaces of points in projective space.

To my parents

# Contents

# List of Figures

# List of Tables

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# Chapter 1

# Introduction

This thesis is divided into two parts, not including the introduction. The first part deals with the amplituhedron, a geometric object arising from computations in particle scattering [AT14]. It is a semialgebraic subset of a real Grassmannian, whose boundaries encode the poles of certain rational functions appearing as scattering amplitudes in physics. One special case is the positive Grassmannian, defined as the locus in the real Grassmannian where the Plücker coordinates are positive. This semialgebraic set appears in mathematics and nature, with connections to combinatorics, physics, electrical networks, and many other places. The first two chapters contribute to the theory of amplituhedra, based on work with Dmitrii Pavlov and Yelena Mandelshtam [MPP25] and Elia Mazzucchelli [MP25].

At the end of the first part, we discuss positive geometry. This is an emerging branch of math which studies semialgebraic subsets of real algebraic varieties defined by natural positivity conditions. Again, a prominent example is the positive Grassmannian as a subset of the real Grassmannian. Another special case is the moduli space of $n$ labeled points on the real projective line, where the positive locus parameterizes configurations in which the points are arranged in cyclic order. In Chapter 4, based on joint work with Sarah Brauner, Chris Eur, and Raluca Vlad, we provide a new family of positive geometries called *wondertopes* which generalize this latter case [Bra+25].

Finally, the second part of this thesis explains some connections of this work to classical algebraic geometry. In particular, we generalize the classical *Chow form* of a projective variety to the *Chow–Lam form*, which is defined for subvarieties of arbitrary Grassmannians. We then prove that certain polynomials appearing in scattering amplitudes are Chow–Lam forms of positroid varieties. The Chow–Lam form is also of intrinsic interest in computational algebraic geometry, and we provide applications to computer vision and to configuration spaces of points in projective space. Chapters 5–7 are based on the papers [PS25] with Bernd Sturmfels and [PR25] with Kristian Ranestad, and the single-author paper [Pra25].

The following subsections provide background and motivation for this work. At the end of each subsection, we summarize our key contributions. Aside from the introduction, the two parts may be read independently.

## 1.1 Scattering amplitudes and the amplituhedron

A major goal of high-energy physicists is to predict the outcome of particle collisions. Given a collection of incoming and outgoing particles with fixed momenta, the key value to compute is the *scattering amplitude* of the particle configuration. This is a function of the momenta from which one may extract scattering probabilities through an additional integration step. Theoretical physicists will predict this scattering amplitude, and pass it along to experimental physicists working at a large particle collider; in this way they may determine if their theories match reality, and predict the existence of new particles.

Since the 1940s, the best way to compute these amplitudes has been via Feynman diagrams. Feynman's idea was the following: fix a collection of incoming and outgoing particles, and consider all the directed graphs that can lie in between. Each such graph represents a potential particle interaction, and is called a Feynman diagram in this context. One associates to each graph an integral, and sums over all integrals to obtain the amplitude.

Of course, there are infinitely many diagrams, but the contributions of each diagram get smaller as the number of loops in the graph increases. This infinite sum is believed to converge in cases of interest, e.g. for the theory of *gluons in $\mathcal{N} = 4$ super Yang-Mills*. In practice even the *tree-level scattering amplitude*, obtained by considering only diagrams which are trees, is valuable to compute. For a thorough treatment of Feynman integrals and their computation, see [Wei22].
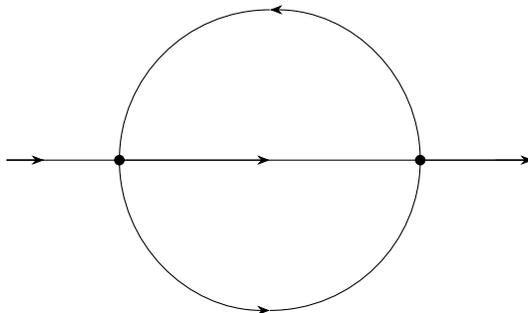


Figure 1.1: A Feynman graph with one incoming and one outgoing particle

However, Feynman's method involves large amounts of cancellation and spurious singularities. For example, the tree-level scattering amplitude for two-gluon to four-gluon scattering involves hundreds of pages of calculations when done via Feynman diagrams, while the end result is a one-line rational function [PT86]. For a while, there was no satisfying theoretical explanation for this simple final result, and the massive cancellation remained a mystery. A breakthrough came in 2013 due to Nima Arkani-Hamed and Jaroslav Trnka, who showed this rational function may be viewed as a function on the Grassmannian with poles exactly along the boundary of a certain semialgebraic set. They called this set the *amplituhedron*. This discovery inspired a flurry of activity from physicists and mathematicians alike, in an effort to understand this new object and its place within mathematics.

To define the amplituhedron, we first need to discuss the Grassmannian. The *real Grassmannian* is a real algebraic variety which parameterizes $k$-dimensional subspaces of $\mathbb{R}^n$. The *nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k, n)$ is the subset of $\mathrm{Gr}(k, n)$ where the Plücker coordinates are nonnegative (or indeed, where they all have the same sign, since coordinates in projective space are only defined up to common scalar). The positive Grassmannian was first defined by Lusztig in the general context of positive parts of reductive groups [Lus94]. The combinatorics of its boundary has been extensively studied by Postnikov, who established bijections between the boundary components and many different combinatorial objects [Pos06]. More generally, fix $k, m, n$ with $n \geq k + m$ and a real $(k + m) \times n$ matrix $Z$ with positive maximal minors.

**Definition 1.1.1.** The *amplituhedron* $\mathcal{A}_{k,m,n}(Z)$ for $n \geq k + m$ is the image of the linear projection

$$\wedge^k Z \,:\, \mathrm{Gr}_{\geq 0}(k, n) \to \mathrm{Gr}(k, k + m)$$
$$[M] \mapsto [MZ^T], \tag{1.1}$$

where $[M]$ denotes the rowspan of the $k \times n$ matrix $M$.

The case $n = k + m$, with $Z$ the identity matrix, recovers the nonnegative Grassmannian.

**Example 1.1.2** (The case $k = 1$). Here the nonnegative Grassmannian equals the nonnegative orthant in $\mathbb{P}^{n-1}_{\mathbb{R}}$, which may be seen as a simplex in the affine chart where the sum of the coordinates is nonzero. Explicitly, in that chart we have $x_1 + \ldots + x_n = 1$ and $x_i \geq 0$.

Let $Z_1, \ldots, Z_n \in \mathbb{R}^{m+1}$ denote the columns of $Z$. We may view these as points in $\mathbb{P}^m_{\mathbb{R}}$. The amplituhedron $\mathcal{A}_{1,m,n}(Z) \subset \mathbb{P}^m_{\mathbb{R}}$ is then the convex hull of the $n$ points $Z_1, \ldots, Z_n$. Since $Z$ is totally positive, the amplituhedron has the combinatorial type of a cyclic polytope; see the remarks after Lemma 3.3.5 for a proof. Here by *combinatorial type* of a polytope, we mean the lattice of faces of the polytope, ordered by containment.

For example, $\mathcal{A}_{1,2,4}$ is a projection of a tetrahedron to $\mathbb{P}^2$. The positivity of $Z$ guarantees that the image will be a quadrilateral, rather than a triangle with a point in the middle. ◇
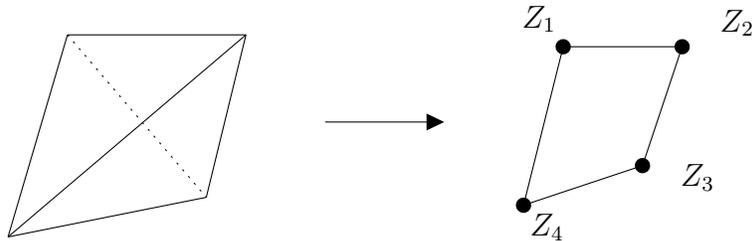


Figure 1.2: The amplituhedron for $k = 1, m = 2, n = 4$

It follows from the Tarski-Seidenberg theorem [BCR10, Section 1.4] that the amplituhedron is a semialgebraic set. The complexity of the amplituhedron is, roughly speaking, controlled by the value of $m$. The special case $m = 4$ is the one relevant to physics [AT14].

In the cases $m = 1, 2$, and 4, the boundaries of the amplituhedron are particularly nice; they are certain linear expressions in the Plücker coordinates of $\mathrm{Gr}(k, k + m)$, which can be conveniently written using *twistor coordinates*. These were first introduced by Hodges [Hod13] in a paper on cyclic polytopes, which inspired the later amplituhedron paper.

**Definition 1.1.3** (Twistor coordinates). Let $Z$ be a real $(k + m) \times n$ matrix of full rank. Let $Y$ be a $k \times k + m$ matrix of full rank, whose rowspan $[Y]$ is a point in $\mathrm{Gr}(k, k + m)$. The *twistor coordinates* of $[Y]$ with respect to $Z$ are defined as

$$\langle Y\, i_1 \ldots i_m \rangle := \det \begin{bmatrix} | & & | & \\ Z_{i_1} & \ldots & Z_{i_m} & Y^T \\ | & & | & \end{bmatrix}, \tag{1.2}$$

for each tuple $1 \le i_1 < \ldots < i_m \le n$.

One may recover the ordinary Plücker coordinates from (1.2) via Laplace expansion of the determinant. Furthermore, twistor coordinates satisfy the Plücker relations for $\mathrm{Gr}(m, n)$.

**Example 1.1.4** (Twistor coordinates on $\mathrm{Gr}(2, 4)$). The real Grassmannian $\mathrm{Gr}(2, 4)$ can be thought of as a parameter space for lines in $\mathbb{P}^3$, via the usual projection map from $\mathbb{R}^4$ to $\mathbb{P}^3_{\mathbb{R}}$. The twistor coordinate $\langle Y12 \rangle$ vanishes on $\mathrm{Gr}(2, 4)$ whenever the line spanned by $Z_1$ and $Z_2$ meets the line $[Y]$ in $\mathbb{P}^3$. Indeed, this will happen if and only if the lines $\overline{Z_1 Z_2}$ and $[Y]$ are coplanar, in which case the determinant of (1.2) will vanish. $\diamond$
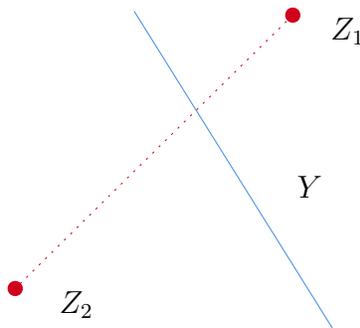


Figure 1.3: A variable line $[Y]$ meeting the fixed line $\overline{Z_1 Z_2}$

The *algebraic boundary* of the amplituhedron is the Zariski closure in the complex Grassmannian $\mathrm{Gr}(k, k + m)$ of the Euclidean boundary. The following results are known about the algebraic boundary of the amplituhedron.

- The algebraic boundary of the $m = 1$ amplituhedron is given by $\langle Y1 \rangle, \ldots, \langle Yn \rangle = 0$ [KW19].

- The algebraic boundary of the $m = 2$ amplituhedron is given by $\langle Y12 \rangle, \ldots, \langle Y\,n-1\,n \rangle$, $\langle Y1n \rangle = 0$ [RST24, Proposition 3.1].

- The algebraic boundary of the $m = 4$ amplituhedron is given by $\langle Y\,i\,i+1\,j\,j+1 \rangle = 0$, for $1 \leq i < j \leq n$ [ELT25, Corollary 8.8].

- For $m = 6$ (and conjecturally for higher $m$), the algebraic boundary of the amplituhedron involves some higher-degree polynomials in twistor coordinates [Eve+25].

In Chapter 2, we study a generalization of amplituhedra called *Grasstopes*. This is based on the paper [MPP25] with Dmitrii Pavlov and Yelena Mandelshtam. Grasstopes are defined in the same manner as amplituhedra, but without the positivity assumption on the matrix $Z$. We focus in particular on the $m = 1$ case, where we obtain a complete characterization, generalizing the work of Karp and Williams [KW19]. In this case the amplituhedron lives in the Grassmannian $\mathrm{Gr}(k, k+1)$, which is isomorphic to projective space $\mathbb{P}^k$. Our main theorem is that, as for the amplituhedron, Grasstopes are unions of certain regions in the complement of the hyperplane arrangement given by the vanishing of $\langle Y1 \rangle, \ldots, \langle Yn \rangle$.

We also use our theorem to give examples of Grasstopes with interesting topology. While $m = 1$ amplituhedra are homeomorphic to closed balls [KW19], we will show that $m = 1$ Grasstopes may be non-orientable.

**Example 1.1.5.** This is Example 2.3.3 from Chapter 2. The matrix in question is

$$
Z = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & 0 & -2 & -1 \end{bmatrix}^T.
$$

The Grasstope is the image of the positive Grassmannian under the map (1.1), and is depicted as the shaded region in Figure 1.4. It consists of regions whose sign vectors have at least two sign changes. It is the complement of a disc in $\mathbb{P}^2_{\mathbb{R}}$, so it is homeomorphic to a Möbius strip. ◇

In Chapter 3, following the paper [MP25] with Elia Mazzucchelli, we study the convex hull of the amplituhedron in its ambient Plücker space. This is a polytope, which we call the *exterior cyclic polytope* $C_{k,m,n}(Z)$. It generalizes the cyclic polytope from Example 1.1.2. Our first main theorem in Chapter 3 is the following.

**Theorem 3.5.7.** We have that

$$
\mathcal{A}_{2,2,n}(Z) = \mathrm{Gr}(2, 4) \cap C_{2,2,n}(Z), \tag{1.3}
$$

for every real $4 \times n$ matrix $Z$ with positive maximal minors.
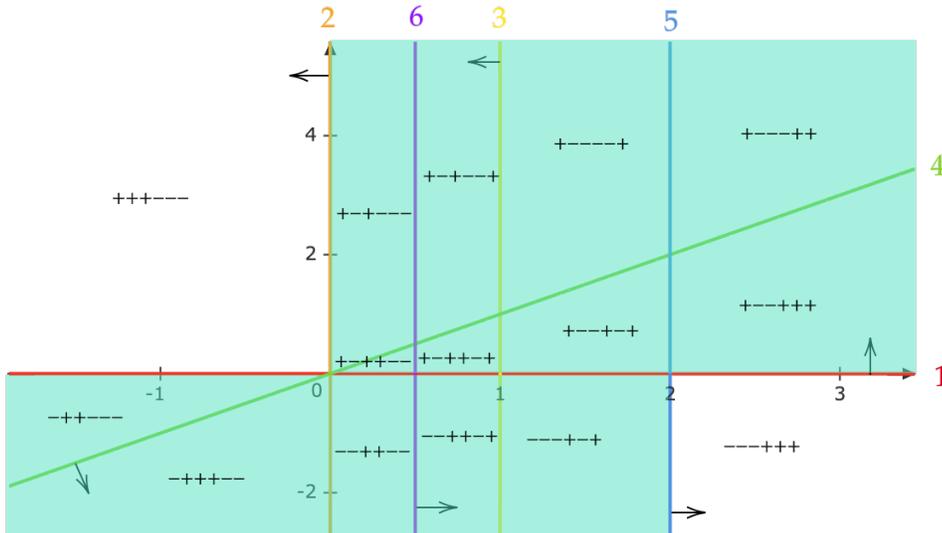
Figure 1.4: A non-orientable Grasstope. The complement is a disc in $\mathbb{P}^2_{\mathbb{R}}$.

We also use our description of $C_{k,m,n}(Z)$ and its dual to give a *dual amplituhedron* in the case $k = m = 2$. A priori, there is no definition of the dual of a semialgebraic subset inside the real Grassmannian. Our definition extends the notion of the polar dual of a convex set in $\mathbb{R}^n$ to a connected semialgebraic subset of any embedded projective variety.

## 1.2 Positive geometry

Over the past few years, there has been much effort to define a mathematical concept which captures features of the positive Grassmannian and the amplituhedron. The paper [ABL17] was the first to put forth a formal definition of a *positive geometry*. In their language, a positive geometry is a triple of a complex projective variety $X$, a semialgebraic subset $X_{\geq 0}$ of its real locus, and a meromorphic top-form $\Omega$, satisfying the conditions of Definition 1.2.3. To state these conditions precisely, we first introduce the *Poincaré residue*, which extends the residue from single-variable complex analysis to the multivariate setting.

Let $X$ be an irreducible complex $n$-dimensional algebraic variety, $\Omega$ a meromorphic $n$-form on $X$, and $H \subset X$ an irreducible hypersurface on $X$. Assume that $\Omega$ has at most simple poles on $H$. Let $f$ be a local coordinate such that $f$ vanishes to order one on $H$. Write

$$\Omega = \frac{df}{f} \wedge \eta + \eta'$$

for an $(n-1)$-form $\eta$ and a $n$-form $\eta'$, both without poles along $H$.

**Definition 1.2.1.** The *(Poincaré) residue* is the $(n-1)$-form on $H$ given by

$$\operatorname{Res}_H \Omega := \eta\big|_H. \tag{1.4}$$

In particular, if $\Omega$ does not have a pole along $H$, then the residue is zero.

This residue does not depend on the choices of $f$, $\eta$, and $\eta'$. To compute this decomposition concretely, one must pass to a system of local coordinates, as in Examples 1.2.2 and 1.2.4.

**Example 1.2.2.** Let $X = \mathbb{P}^1$. Let $\Omega$ be the differential form given in local coordinates by

$$\Omega = \operatorname{dlog} \frac{x-b}{x-a} = \frac{b-a}{(x-a)(x-b)} dx.$$

Then the residue at the point $a$ is $\frac{b-a}{x-b}\big|_{x=a} = -1$, and similarly the residue at $b$ is 1. $\diamond$

**Definition 1.2.3.** A *positive geometry* is a pair $(X, X_{\geq 0})$ consisting of an $n$-dimensional irreducible normal complex projective variety $X$ defined over $\mathbb{R}$, and a semialgebraic subset $X_{\geq 0} \subset X(\mathbb{R})$, along with a unique nonzero meromorphic $n$-form $\Omega(X, X_{\geq} 0)$, satisfying the following properties.

- The interior $X_{>0} := \operatorname{int}(X_{\geq 0})$ of $X_{\geq 0}$ is a smooth oriented real $n$-manifold, and the Euclidean closure of $X_{>0}$ is $X_{\geq 0}$.

- Let $\partial X$ be the Zariski closure of $X_{\geq 0} \setminus X_{>0}$, whose codimension 1 irreducible components are $C_1, \ldots, C_k$. Then, with $C_{i, \geq 0}$ as the Euclidean closure of $\operatorname{int}(C_i \cap (X_{\geq 0} \setminus X_{>0}))$ in $C_i(\mathbb{R})$, the $n$-form $\Omega(X, X_{\geq 0})$ satisfies the following recursive property:

  (a) if $n = 0$, then $X = X_{\geq 0}$ is a point and $\Omega(X, X_{\geq 0}) = \pm 1$, depending on orientation; and

  (b) if $n > 0$, the form $\Omega(X, X_{\geq 0})$ has simple poles along each $C_i$ and no other poles, and every $(C_i, C_{i, \geq 0})$ is a positive geometry, with the orientation on $C_{i, >0}$ inherited from that of $X_{>0}$, such that the residue of $\Omega(X, X_{\geq 0})$ along $C_i$ satisfies

$$\operatorname{Res}_{C_i} \Omega(X, X_{\geq 0}) = \Omega(C_i, C_{i, \geq 0}).$$

The $n$-form $\Omega(X, X_{\geq 0})$ is called the *canonical form* of $(X, X_{\geq 0})$, and $X_{\geq 0}$ the *nonnegative part*. We call $(C_i, C_{i, \geq 0})$ a (codimension 1) *boundary component* of $(X, X_{\geq 0})$.

Many well-studied spaces are positive geometries with naturally arising nonnegative regions. Examples include flag varieties and toric varieties [ABL17], the moduli space $\overline{M}_{0,n}$ of $n$-pointed stable rational curves [AHL21a], and del Pezzo surfaces and their moduli spaces [Ear+25]. Conjecturally, the amplituhedron is the nonnegative region of a positive geometry,

where the complex manifold is a Grassmannian [ABL17, Conjecture 6.3]. For an introduction to the field, see [Lam24; RST25]. The prototypical example of a positive geometry is $(\mathbb{P}^n, \mathcal{P})$ where $\mathcal{P}$ is a polytope in real projective space [ABL17, Section 6]. The following example gives an explicit computation of the canonical form of a particular quadrilateral.

**Example 1.2.4** (Quadrilateral in $\mathbb{P}^2$)**.** Consider the quadrilateral $\mathcal{P}$ in Figure 1.5, whose bounding hyperplanes are given by linear forms $x$, $y$, $x+2y-2$, and $2x+y-2$. Its canonical
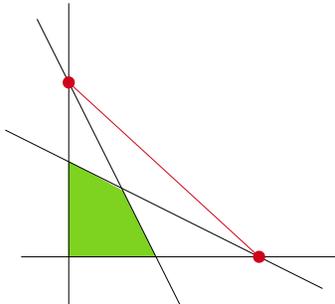


Figure 1.5: A quadrilateral (green) and its adjoint line (red)

form is
$$\Omega_{\mathcal{P}} = \frac{x+y-1}{xy(x+2y-2)(2x+y-2)}dxdy.$$
The numerator is needed to cancel out spurious poles, so that the residues are correct. For example, the residue along the boundary component $\{x = 0\}$ is

$$\mathrm{Res}_{\{x=0\}} \Omega_{\mathcal{P}} = \mathrm{Res}_{\{x=0\}} \frac{dx}{x} \wedge \frac{x+y-1}{y(x+2y-2)(2x+y-2)}dy$$
$$= \frac{x+y-2}{y(x+2y-2)(2x+y-2)}dy\Big|_{\{x=0\}} = \frac{y-2}{y(2y-2)(y-2)}dy = \frac{1}{y(y-1)}dy.$$

The latter has poles only at $y = 0$ and $y = 1$, reflecting the fact that the point $(0, 2)$ is not a face of the polytope. The numerator of the canonical form is called the *adjoint* of the polytope. For more on computing canonical forms of polytopes, see [Lam24]. $\diamond$

Chapter 4 introduces a new family of positive geometries called *wonderful polytopes*, or *wondertopes*, which generalize the case of a polytope in projective space. It is based on joint work with Sarah Brauner, Chris Eur, and Raluca Vlad [Bra+25].

Let $V \cong \mathbb{R}^n$ be a real vector space. We will slightly abuse notation and use $\mathbb{P}V$ to denote either its projectivization or the projectivization of $\mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C})$, whenever the underlying field is clear from context. A wondertope is constructed from a polytope $\mathcal{P} \subset \mathbb{P}V$ and a set $\mathcal{B}$ of proper linear subvarieties of $\mathbb{P}V$ satisfying the following properties:

(1) for every $F \in \mathcal{B}$, the intersection $F \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$,

(2) every hyperplane whose intersection with $\mathcal{P}$ is a facet of $\mathcal{P}$ is in $\mathcal{B}$, and

(3) $\mathcal{B}$ is a *building set* in the sense of [DP95, Theorem 2.3.(2)]; that is, for every intersection $L = \bigcap_{F \in \mathcal{F}} F$ of a subset $\mathcal{F}$ of $\mathcal{B}$, the subset

$$\mathcal{F}_L = \{F \in \mathcal{B} \mid F \text{ is minimal (by inclusion) among elements of } \mathcal{B} \text{ that contain } L\}$$

of $\mathcal{B}$ satisfies $\displaystyle\sum_{F \in \mathcal{F}_L} \mathrm{codim}_{\mathbb{P}V} F = \mathrm{codim}_{\mathbb{P}V} L$.

**Definition 1.2.5.** With notation as above, choose any ordering of $\mathcal{B} = \{F_1, \ldots, F_k\}$ such that $i \leq j$ if $F_i \subseteq F_j$, and define $X^{\mathcal{B}}$, also denoted $\mathbb{P}V^{\mathcal{B}}$, to be the sequential blow-up

$$X^{\mathbb{F}} := \mathrm{Bl}_{F_k}(\cdots(\mathrm{Bl}_{F_2}(\mathrm{Bl}_{F_1} \mathbb{P}V))\cdots)$$

with the blow-down map $\pi_{\mathbb{F}} : X^{\mathbb{F}} \longrightarrow \mathbb{P}V$. Here, we reuse notation to write $F_2$ for the strict transform of $F_2$ in $\mathrm{Bl}_{F_1} \mathbb{P}V$, and similarly for $F_3, \ldots, F_k$. The *wondertope* $\widetilde{\mathcal{P}}^{\mathcal{B}}$ is defined as

$$\widetilde{\mathcal{P}}^{\mathcal{B}} = \text{the Euclidean closure of } \pi_{\mathcal{B}}^{-1}(\text{the interior of } \mathcal{P}) \text{ in } X^{\mathcal{B}}(\mathbb{R}).$$

The variety $X^{\mathcal{B}}$ is known as the *wonderful compactification* of the arrangement complement $C(\mathcal{B}) = \mathbb{P}V \setminus (\bigcup \mathcal{B})$, introduced by De Concini and Procesi in [DP95]. That a wondertope is semialgebraic is verified in Corollary 4.2.6. Our main result is the following.

**Theorem 1.2.6.** *The pair $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ is a positive geometry. Its canonical form is the pullback $\pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P})$ of the canonical form of $(\mathbb{P}V, \mathcal{P})$. The boundary components of $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ are (the strict transforms of) the exceptional divisors $E_F$ for $F \in \mathcal{B}$ such that $\dim(\mathcal{P} \cap F) = \dim F$.*

One special case of Theorem 1.2.6 arises from the *rank $(n-1)$ braid arrangement* $A_{n-1}$ comprised of hyperplanes $\{x_i - x_j = 0\}$ in $V = \mathbb{R}^n / \mathbb{R} \cdot (1, 1, \ldots, 1)$ for $1 \leq i < j \leq n$. The complement $C(A_{n-1}) = \mathbb{P}V \setminus A_{n-1}$ is isomorphic to $M_{0,n+1}$, the moduli space of $n + 1$ distinct points on $\mathbb{P}^1$. For an appropriate choice of building set, the corresponding wonderful compactification is isomorphic to the Deligne–Knudsen–Mumford compactification $\overline{M}_{0,n+1}$ [DM69]. The polytope in this context is a simplex, and the corresponding wondertope $(\overline{M}_{0,n+1})_{\geq 0}$ is combinatorially isomorphic to the $n$-associahedron, which parameterizes triangulations of an $(n + 1)$-gon. It is sometimes called the "curvy associahedron." The systematic study of $(\overline{M}_{0,n+1})_{\geq 0}$ was initiated by Devadoss [Dev99].

More recently, Brown and Dupont have developed a new framework for positive geometry based on Deligne's mixed Hodge theory [BD25]. Rather than a semialgebraic set, their main object of interest is a pair $(X, Y)$ where $X$ is a complex projective variety, $Y \subset X$ is a closed subvariety, and the complement $X \setminus Y$ is smooth. Canonical forms are recast as a map from $H_n(X, Y)$ to the vector space $\Omega_{\log}^n(X \setminus Y)$ of holomorphic top-forms on $X \setminus Y$ with logarithmic poles at infinity. In their setting, invariance under blow-ups is immediate,
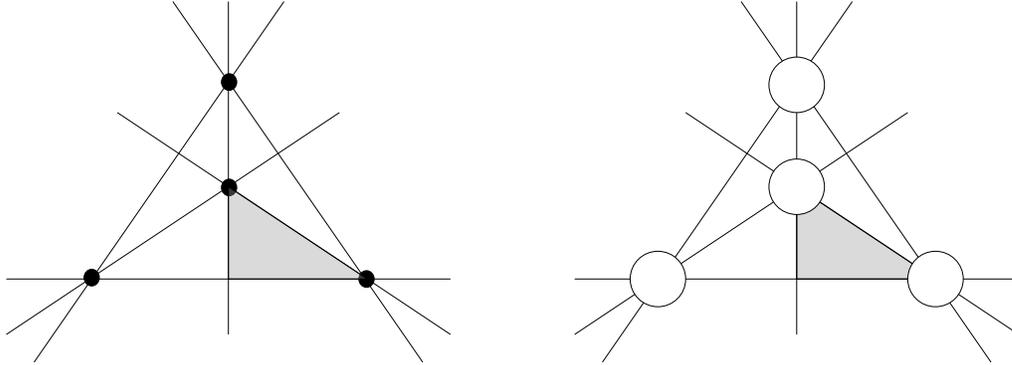
Figure 1.6: A region (a simplex) in $M_{0,5} \cong C(A_3)$ in $\mathbb{P}^2$ (left) and its wondertope $(\overline{M}_{0,5})_{\geq 0}$ (an associahedron) in $\overline{M}_{0,5}$ (right).

whereas Chapter 4 will show that this is not the case for Definition 1.2.3 (see non-examples in Sections 4.5.2 and 4.5.3). While we will not use their theory within this thesis, we believe that it is a promising new viewpoint which will drive further development in the subject.

We reiterate that, while these definitions were inspired by the amplituhedron and surrounding mathematics, it is still unknown whether amplituhedra are positive geometries in general. The only known cases are $\mathcal{A}_{2,2,n}(Z) \subset \mathrm{Gr}(2,4)$, in which case the conjecture was proven by explicit analysis of the boundary components and their intersections [RST24].

## 1.3 Classical algebraic geometry

It is a fact of life that any algebraic curve in $\mathbb{P}^2$ can be represented as the zero set of a single irreducible polynomial. In 1864, Cayley asked the question: can a curve in $\mathbb{P}^3$ be captured by a single irreducible polynomial? Of course, one needs at least two polynomials to cut it out of the ambient space. But Cayley came up with the following trick: consider the set of lines in $\mathbb{P}^3$ which meet a given curve $\mathcal{V}$. These lines form a hypersurface in $\mathrm{Gr}(2,4)$, whose points parameterize lines in $\mathbb{P}^3$. This hypersurface is defined by a single equation, called the *Cayley form* of the curve [Cay62]. This is the desired polynomial. Indeed it is a nontrivial fact that, given the Cayley form, one can recover the radical ideal of the curve.

**Example 1.3.1** (Twisted cubic)**.** Let $\mathcal{V}$ be the closure of $(1 : t : t^2 : t^3)$ in $\mathbb{P}^3$. Its Cayley form in primal Plücker coordinates is the determinant of the *Bézout matrix*:

$$\mathrm{C}_{\mathcal{V}} \;=\; \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}. \tag{1.5}$$

Passing to coordinates $p_{ij} = a_i b_j - a_j b_i$, we obtain the *Sylvester resultant*

$$\mathrm{C}_{\mathcal{V}} \;=\; \mathrm{Res}_t\big(a_1 + a_2 t + a_3 t^2 + a_4 t^3, \, b_1 + b_2 t + b_3 t^2 + b_4 t^3\big).$$

This is a polynomial of degree 6 in 8 unknowns. It characterizes pairs of cubics in one variable with a common root. ◇

The Cayley form was generalized in the 1930s by Chow and van der Waerden to any projective variety [CW37]. That is, take a variety $\mathcal{V} \subset \mathbb{P}^{n-1}$ of dimension $d$ and consider

$$\mathcal{C}_\mathcal{V} := \{L \in \mathrm{Gr}(n-d-1, n) \ : \ L \cap \mathcal{V} \neq \varnothing\}.$$

The homogeneous coordinate ring of the Grassmannian is a UFD, so $\mathcal{C}_\mathcal{V}$ is cut out by a single equation $\mathrm{C}_\mathcal{V}$ in Plücker coordinates. We call $\mathrm{C}_\mathcal{V}$ the *Chow form* of $\mathcal{V}$ [CW37].

Chow forms have many nice properties. If $\mathcal{V}$ is irreducible, so is its Chow form. The degree of the Chow form equals the degree of the original variety. Finally, given $\mathrm{C}_\mathcal{V}$, one can reconstruct the radical ideal of $\mathcal{V}$ [Cat92]. This last property was the original motivation for Chow and van der Waerden's construction.

In Chapter 5, we generalize the Chow form to certain subvarieties of an arbitrary Grassmannian. Suppose we are given a subvariety $\mathcal{V}$ of $\mathrm{Gr}(k, n)$, where $\dim(\mathcal{V}) = k(r-k) - 1$ for some $r \in \{k+1, \ldots, n\}$. Our aim is to characterize $\mathcal{V}$ by a single equation. Let $\mathcal{CL}_\mathcal{V}$ denote the set of linear spaces $P \in \mathrm{Gr}(k+n-r, n)$ which contain a subspace $Q$ belonging to $\mathcal{V}$. The codimension of $\mathcal{CL}_\mathcal{V}$ in $\mathrm{Gr}(k+n-r, n)$ is expected to be one. If it is one, then $\mathcal{CL}_\mathcal{V}$ is defined by a homogeneous polynomial in Plücker coordinates, which is unique modulo Plücker relations. This polynomial is denoted $\mathrm{CL}_\mathcal{V}$ and called the *Chow–Lam form* of $\mathcal{V}$. For $k \in \{1, n-1\}$, this specializes to the Chow form of a projective variety $\mathcal{V} \subset \mathbb{P}^{n-1}$.

**Example 1.3.2.** Suppose that $n = r$, in which case $\mathcal{V}$ is already a hypersurface in its ambient Grassmannian. In this case its Chow–Lam form is just its defining equation. ◇

**Example 1.3.3** (Ruled surface in $\mathbb{P}^3$)**.** Let $\mathcal{V}$ be a curve in the Grassmannian $\mathrm{Gr}(2, 4)$. Here $k = 2, n = 4$, and $r = 3$. Thus the Chow–Lam locus lives in $\mathrm{Gr}(3, 4) = (\mathbb{P}^3)^\vee$. It consists of planes containing a line in $\mathbb{P}^3$, such that the line is a point of $\mathcal{V}$.
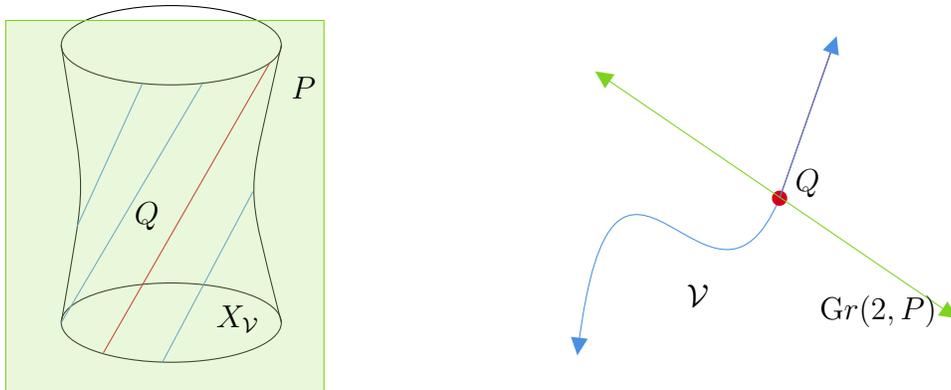


Figure 1.7: Geometry in $\mathbb{P}^3$ (left) and $\mathrm{Gr}(2, 4)$ (right)

Let $X_\mathcal{V}$ be the ruled surface in $\mathbb{P}^3$ swept out by all of the lines in $\mathcal{V}$. Then any tangent plane to $X_\mathcal{V}$ contains the two lines in $X_\mathcal{V}$ through the point of tangency; thus the dual surface $X_\mathcal{V}^\vee$ is contained in $\mathcal{CL}_\mathcal{V}$. In fact, since they are irreducible varieties of the same dimension, we have that the Chow–Lam locus is precisely $X_\mathcal{V}^\vee$. $\diamond$

This generalization to Grassmannians is inspired by computations in positive geometry. Indeed, the theory of Chow–Lam forms is closely connected to the map (1.1). The following theorem tells us that when $\wedge^k Z(\mathcal{V})$ is a hypersurface, we can read its defining equation from the Chow–Lam form of $\mathcal{V}$. The idea is to project to $\mathrm{Gr}(k, r)$, at which point the projection is a hypersurface, and its Chow–Lam form is simply its defining equation.

**Theorem 5.2.10.** The Chow–Lam form $\mathrm{CL}_{\wedge^k Z(\mathcal{V})}$ is obtained from the Chow–Lam form $\mathrm{CL}_\mathcal{V}$ by replacing each primal Plücker coordinate $p_{i_1 \dots i_k}$ with the twistor coordinate $\langle Y\, i_1 \dots i_k \rangle$, for $1 \leq i_1 < \dots < i_k \leq n$.

**Example 1.3.4.** The twisted cubic is the embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$ given by $[s : t] \mapsto [s^3 : s^2 t : s t^2 : t^3]$. Suppose that $Z$ is any $3 \times 4$ matrix of full rank. Then the equation defining its projection to $\mathbb{P}^2$ via $Z$ can be read off from (5.2) by replacing $p_{ij}$ with $\langle Y ij \rangle = \det[Y\, Z_i\, Z_j]$, where $Y$ is a $1 \times 3$ matrix representing a point in $\mathbb{P}^2$. For example,

$$\langle Y 12 \rangle = (z_{1i} z_{2j} - z_{2i} z_{1j}) y_3 - (z_{1i} z_{3j} - z_{3i} z_{1j}) y_2 + (z_{2i} z_{3j} - z_{3i} z_{2j}) y_1.$$

Thus (5.2) functions as the universal equation of the projected curve in $\mathbb{P}^2$, capturing the data of all linear projections at once. $\diamond$

The name "Chow–Lam form" recognizes work of Thomas Lam [Lam16b; Lam16a] at the interface of combinatorics and particle physics. Lam focuses on the case when $\mathcal{V}$ is a positroid variety, namely a boundary component of the positive Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$. Concretely, a positroid variety is defined by the vanishing of certain collections of Plücker coordinates. Lam refers to $\mathcal{CL}_\mathcal{V}$ as the *universal amplituhedron variety* [Lam16b, Section 18.1], and he discusses universal projections via twistor coordinates, as in Proposition 5.2.10.

The three chapters in Part II do the following.

- Chapter 5: we explain the basic properties of Chow–Lam forms, such as degree, and show how to compute them in many practical examples. This is based on the paper [PS25] with Bernd Sturmfels.

- Chapter 6: One can completely recover a variety from its Chow form. We explore to what extent this holds for Chow–Lam forms, and give many interesting families where recovery is not possible. This is based on the paper [PR25] with Kristian Ranestad.

- Chapter 7: Chow forms of toric varieties have particularly rich combinatorics. We characterize Chow–Lam forms of torus orbit closures in the Grassmannian. We also discuss applications of the Chow–Lam form to algebraic vision and to the moduli space of point configurations on the projective line.

The introduction and study of Chow–Lam forms is part of a larger pattern of discriminants appearing in physics, not just in the study of the amplituhedron but also in the study of the individual Feynman integrals [24]. Developing the theory of these discriminants systematically is useful for mathematics and physics alike.

# Part I

# Scattering amplitudes and the amplituhedron

# Chapter 2

# Grasstopes

This chapter is based on the paper *Combinatorics of $m = 1$ Grasstopes*, which is joint work with Dmitrii Pavlov and Yelena Mandelshtam [MPP25]. We give new proofs of the main theorems, the original versions of which can be found in Sections 2-3 of the aforementioned paper. We also added Example 2.3.4. The material in Sections 4-6 is largely unchanged.

As previously mentioned, the amplituhedron is a semialgebraic set whose boundary encodes certain scattering amplitudes in physics. In recent years, the amplituhedron has been studied extensively from the point of view of algebraic combinatorics for $m = 1, 2, 4$ (see [ELT25; GL20; KM23; KW19; LPW23; PSW23]). While the $m = 4$ case is the most physically relevant, the other cases have rich combinatorics in their own right and can give intuition towards the $m = 4$ case. The structure of the $m = 1$ amplituhedron is particularly simple: Karp and Williams [KW19] show that it is homeomorphic to the complex of bounded cells of an affine hyperplane arrangement and therefore is homeomorphic to a closed ball.

One reason why the amplituhedron is so amenable to combinatorial study is that the totally nonnegative Grassmannian has a rich combinatorial structure. In particular, $\mathrm{Gr}_{\geq 0}(k, n)$ admits a stratification by positroid cells, which are all homeomorphic to open balls [Pos06]. However, amplituhedra are images of very special linear maps, just as cyclic polytopes are very special polytopes. From this point of view, it makes sense to consider images of the totally nonnegative Grassmannian under arbitrary linear maps. In [Lam14], Lam considers the images of positroid cells under arbitrary linear maps, and calls them *Grassmann polytopes*. Images of the entire totally nonnegative Grassmannian are *full Grassmann polytopes*.

The goal of this chapter is to relax the positivity condition on $Z$ in the definition of the amplituhedron, and see how this affects the topology of the resulting semialgebraic set. The rest of the chapter is organized as follows. In Section 2.1, we define Grasstopes. We introduce the concepts necessary for the sign variation characterization of $m = 1$ Grasstopes and prove some auxiliary results about general Grasstopes. In Section 2.2, we study the combinatorics and geometry of Grasstopes for $m = 1$ and prove the sign variation characterization results for Grasstopes, as well as for open rational Grasstopes. Section 2.3 is devoted to examples. In Section 2.4, we give background on oriented matroids, which is useful for Section 2.5. Finally, in Section 2.5 we investigate how many regions of a hyperplane arrangement can

be in an $m = 1$ Grasstope, and, based on the sign variation characterization, suggest a definition of the Grasstope of a (not necessarily realizable) oriented matroid.

## 2.1 Preliminaries

### 2.1.1 Grassmann polytopes

Let $Z$ be a real $(k + m) \times n$ matrix of full rank, where $k + m \leq n$. The matrix $Z$ defines a rational map

$$\tilde{Z} : \mathrm{Gr}(k, n) \dashrightarrow \mathrm{Gr}(k, k + m)$$
$$[A] \mapsto [AZ^T],$$

where $[A]$ is the class in $\mathrm{Gr}(k, n)$ of a matrix $A$.

**Definition 2.1.1** (Grasstopes)**.** The image $\tilde{Z}(\mathrm{Gr}_{\geq 0}(k, n)) \subseteq \mathrm{Gr}(k, k + m)$ is called the $(k, m, n)$-*Grasstope* of $Z$ and is denoted by $\mathcal{G}_{k,m,n}(Z)$.

In the special case where $Z$ has positive maximal minors, we recover the definition of the amplituhedron $\mathcal{A}_{k,m,n}(Z)$. We remark that the map $\tilde{Z}$ is really the composition of two maps. To see this, let $W$ be the rowspan of the matrix $Z$. Then we have a map

$$\tilde{Z} : \mathrm{Gr}(k, n) \dashrightarrow \mathrm{Gr}(k, W)$$
$$[A] \mapsto \mathrm{Proj}_W([A]),$$

which projects the subspace represented by $A$ to $W$. To obtain the amplituhedron map, we choose an isomorphism of $\mathrm{Gr}(k, W)$ with $\mathrm{Gr}(k, k + m)$. A choice of such an isomorphism is equivalent to a choice of basis for $W$, which is provided by the rows of the matrix $Z$.

Note that the map $\tilde{Z}$ is not everywhere defined. Indeed, the projection of $[A]$ to $W$ may have dimension smaller than $k$, in which case it is not an element of the Grassmannian $\mathrm{Gr}(k, W)$. In general, the map $\tilde{Z}$ has base locus

$$\Omega(\tilde{Z}) := \{V : \dim(V \cap \ker Z) \geq 1\} \subset \mathrm{Gr}(k, n),$$

which may or may not intersect $\mathrm{Gr}_{\geq 0}(k, n)$. The variety $\Omega(\tilde{Z})$ is a Schubert variety, so in particular it is closed in $\mathrm{Gr}(k, n)$ [Lam14, Section 17].

Even if the base locus $\Omega(\tilde{Z})$ has nonempty intersection with the nonnegative Grassmannian, it still makes sense to consider the image of $\mathrm{Gr}_{\geq 0}(k, n) \backslash (\Omega(\tilde{Z}) \cap \mathrm{Gr}_{\geq 0}(k, n))$. Even when the map $\tilde{Z}$ is not regular on the nonnegative Grassmannian, we will still write $\tilde{Z}(\mathrm{Gr}_{\geq 0}(k, n))$ for this image, and call it a *rational Grasstope*.

**Definition 2.1.2** (Rational Grasstope)**.** Suppose the map $\tilde{Z}$ is not well-defined on $\mathrm{Gr}(k, n)_{\geq 0}$. Then the image $\mathcal{G}_{k,m,n}(Z) = \tilde{Z}(\mathrm{Gr}(k, n)_{\geq 0})$ is called a *rational* Grasstope. The image $\mathcal{G}^{\circ}_{k,m,n}(Z) = \tilde{Z}(\mathrm{Gr}(k, n)_{> 0})$ of the totally positive Grassmannian is called an *open rational* Grasstope. For $m = 1$, this is indeed an open set, as shown in Proposition 2.2.7.

So far we have defined the Grasstope in terms of projections. However, we may work with another definition, in terms of intersections. This is based on the following fundamental linear algebra fact, which states that projecting and dualizing is the same as slicing the dual.

**Lemma 2.1.3.** *Suppose that $V$ and $W$ are vector spaces of $\mathbb{R}^n$. Then*

$$W = V^\perp \cap W \oplus \mathrm{Proj}_W(V). \tag{2.1}$$

*Proof.* Suppose that $w$ is in $W$. Then, for any $v \in V$, we have $\langle w, \mathrm{Proj}_W(v) \rangle = \langle w, v \rangle$. Therefore,

$$w \perp \mathrm{Proj}_W(V) \quad \Longleftrightarrow \quad w \perp V.$$

It follows that the orthogonal complement of $\mathrm{Proj}_W(V)$ in $W$ is equal to $V^\perp \cap W$. □

As before, let $Z$ be a real $(k+m) \times n$ matrix of full rank, where $k+m \leq n$. Let $W$ be the rowspan of $Z$. We define the map

$$\hat{Z} : \mathrm{Gr}_{\geq 0}(k, n) \to \mathrm{Gr}(m, W)$$
$$[A] \mapsto [\ker A] \cap W.$$

**Definition 2.1.4** (Primal Grasstopes)**.** The image $\hat{Z}(\mathrm{Gr}_{\geq 0}(k, n)) \subseteq \mathrm{Gr}(m, W)$ is called the *primal $(k, m, n)$-Grasstope* of $W$ and is denoted by $\mathcal{H}_{k,m,n}(W)$. The image $\hat{Z}(\mathrm{Gr}_{>0}(k, n)) \subseteq \mathrm{Gr}(m, W)$ is called the *primal rational$(k, m, n)$-Grasstope* of $W$ and is denoted by $\mathcal{H}^\circ_{k,m,n}(W)$.

In the special case where $W$ has strictly positive Plücker coordinates, $\mathcal{H}_{k,m,n}(W)$ is equal to the $\mathcal{B}$-*amplituhedron* $\mathcal{B}_{k,m,n}(W)$, which appears in work of [KW19].

**Lemma 2.1.5.** *The base loci of the maps $\tilde{Z}$ and $\hat{Z}$ coincide.*

*Proof.* By the decomposition in Lemma 2.1.3, $\mathrm{Proj}_W(V)$ has dimension less than $k$ if and only if $V^\perp \cap W$ has dimension greater than $m$. □

There is a map $f_Z$ from $\mathrm{Gr}(m, W)$ to $\mathrm{Gr}(k, k+m)$, given by sending $V$ to $V^\perp \cap W$, and then composing with the isomorphism from $\mathrm{Gr}(k, W)$ to $\mathrm{Gr}(k, k+m)$. By [KW19, Lemma 3.12], the map $f_Z$ restricts to a homeomorphism from $\mathcal{B}_{k,m,n}(Z)$ onto $\mathcal{A}_{k,m,n}(Z)$. The same proof works for the following proposition.

**Proposition 2.1.6.** *For every real $(k+m) \times n$ matrix $Z$ of full rank, the map $f_Z$ is a homeomorphism from $\mathcal{H}_{k,m,n}(W)$ to $\mathcal{G}_{k,m,n}(Z)$.*

### 2.1.2 Sign variation

We now recall the definition of sign variation from [KW19].

**Definition 2.1.7** (Sign variation). Given a sequence $v$ of $n$ real numbers, let var($v$) be the number of sign changes in $v$ (zeros are ignored). Let $\overline{\text{var}}(v) := \max\{\text{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\}$, i.e. $\overline{\text{var}}(v)$ is the maximum number of sign changes in $v$ after a sign is chosen for each zero component. Note that both var and $\overline{\text{var}}$ are well-defined functions of homogeneous coordinates of a point in $\mathbb{P}^{n-1}$.

**Example 2.1.8.** The vector $v = (1, 0, 0, -1, 1, -1)$ has var($v$) = 3 and $\overline{\text{var}}(v) = 5$. $\diamond$

By the following theorem of Gantmacher and Krein, sign variation gives an alternate description of both the nonnegative Grassmannian and the positive Grassmannian. This version of the theorem appears in the work of Karp and Williams [KW19, Theorem 3.4], and indeed we will use it in a very similar manner as they do.

**Theorem 2.1.9** ([GK50, Theorems V.3, V.7, V.1, V.6]). *Let* $V \in \text{Gr}_{k,n}$.

(i) $V \in \text{Gr}_{k,n}^{\geq 0} \iff \text{var}(v) \leq k-1$ *for all* $v \in V \iff \overline{\text{var}}(w) \geq k$ *for all* $w \in V^{\perp} \setminus \{0\}$.

(ii) $V \in \text{Gr}_{k,n}^{>0} \iff \overline{\text{var}}(v) \leq k-1$ *for all* $v \in V \setminus \{0\} \iff \text{var}(w) \geq k$ *for all* $w \in V^{\perp} \setminus \{0\}$.

Furthermore, [Kar17, Lemma 4.1] states that a vector $v$ has sign variation less than or equal to $k-1$ if and only if there exists a vector space $V \in \text{Gr}_{\geq 0}(k, n)$ containing $v$. We will use this result several times in the next section.

### 2.1.3 Tame and wild

Finding combinatorial conditions for $\tilde{Z}$ to be well-defined on $\text{Gr}_{\geq 0}(k, n)$ has been an active area of research for several years. In [Lam14, Proposition 15.2] Lam proved that if $\tilde{Z}$ is well-defined on $\text{Gr}_{\geq 0}(k, n)$, then $\mathcal{G}_{k,m,n}(Z)$ is closed and connected. In the same proposition, he showed that the following condition is sufficient for $\tilde{Z}$ to be well-defined:

There exists a $(k+m) \times k$ matrix $M$ such that all $k \times k$ minors of $Z^T M$ are positive. (2.2)

For a sketch of the proof, assume that $M$ is the $(k+m) \times k$ matrix obtained by taking the first $k$ columns of $Z$. For any $k \times n$ matrix $A$, let $A'$ be the submatrix obtained by taking the first $k+m$ columns of $A$. Then the Plücker coordinate $p_{1...k}(AZ^T)$ is equal to $\det(A'M)$. By the Cauchy-Binet theorem, this equals the sum of products of minors of $A$ and $M$, where each term is the product of a nonnegative and a positive number, and not all minors of $A$ are zero. Thus $\det(A'M)$ is positive, and there is no $A$ in $\text{Gr}_{\geq 0}(k, n)$ such that all Plücker coordinates of $AZ^T$ are zero.

Geometrically, condition (2.2) means that the element of $\text{Gr}(k+m, n)$ represented by $Z$ contains a totally positive $k$-dimensional subspace, that is, an element of $\text{Gr}_{>0}(k, n)$. Lam also conjectured that this condition is necessary for $\tilde{Z}$ to be well-defined on $\text{Gr}_{\geq 0}(k, n)$. This

conjecture turned out to be false, with a counterexample given by Galashin (see [KW19, Remark 9.3] and Example 2.3.2). A combinatorial criterion was given in [Kar17, Theorem 4.2]. Lam's conjecture gives rise to the following definition.

**Definition 2.1.10** (Tame Grasstope)**.** The Grasstope $\mathcal{G}_{k,m,n}(Z)$ is *tame* if $Z$ satisfies (2.2).

We now give a simple geometric criterion of tameness. Although well-known to specialists in this area, this result seems not to have appeared in the literature yet. In what follows, for the sake of simplicity, we slightly abuse notation and write $\mathcal{G}_{k,m,n}(Z)$ both for the Grasstope as a subset of $\mathrm{Gr}(k, k + m)$ and its image under the Plücker embedding.

**Proposition 2.1.11.** *The Grasstope $\mathcal{G}_{k,m,n}(Z)$ is tame if and only if there exists a hyperplane in $\mathbb{P}^{\binom{k+m}{k}-1}$ which does not intersect $\mathcal{G}_{k,m,n}(Z)$.*

*Proof.* Any $(k+m) \times k$ matrix $M$ defines a hyperplane in $\mathbb{P}^{\binom{k+m}{k}-1}$ by its Plücker coordinates: a point $p \in \mathbb{P}^{\binom{k+m}{k}-1}$ lies on the linear subspace defined by $M$ if and only if $(\wedge_k M)^T p = 0$. Suppose that $Z$ satisfies (2.2) and $M$ is a $(k+m) \times k$ matrix such that $Z^T M$ is totally positive. Suppose that a point in $\mathcal{G}_{k,m,n}(Z)$ lies on the hyperplane defined by $M$. Then, for some $A \in \mathrm{Gr}_{\geq 0}(k, n)$, it holds that $(\wedge_k A Z^T)(\wedge_k M) = 0$, which implies that $(\wedge_k A)(\wedge_k(Z^T M)) = 0$. Since all $k \times k$ minors of $A$ are nonnegative, and at least one is nonzero, it is not possible for all $k \times k$ minors of $Z^T M$ to have the same sign. Thus, if there exists $M$ such that $Z^T M$ has all positive (or negative) $k \times k$ minors, then the image $\mathcal{G}_{k,m,n}(Z)$ does not intersect the hyperplane defined by $M$.

Now suppose that $Z$ does not satisfy (2.2), that is, for any $M$ the matrix $Z^T M$ has either a zero $k \times k$ minor or at least one positive and one negative $k \times k$ minor. We will show that there exists a matrix $A \in \mathrm{Gr}_{\geq 0}(k, n)$ such that $(\wedge_k A)(\wedge_k(Z^T M)) = 0$, so that the hyperplane defined by $M$ intersects $\tilde{Z}(\mathrm{Gr}_{\geq 0}(k, n))$.

In the first case, when $Z^T M$ has a zero minor in the $i^{\text{th}}$ Plücker coordinate, one can find an element $A \in \mathrm{Gr}_{\geq 0}(k, n)$ which has all Plücker coordinates equal to zero except for the $i^{\text{th}}$ one. In this case, $(\wedge_k A)(\wedge_k(Z^T M)) = 0$.

Now consider the second case, in which $ZM$ has at least one positive and one negative $k \times k$ minor. By the pigeonhole principle, there exists a set of column indices $I = \{i_1, \ldots, i_{k-1}\}$ such that two Plücker coordinates involving $I$ (which we label $p_{I \cup \{i\}}$ and $p_{I \cup \{j\}}$) have different signs. Then, take $(q_{1,\ldots k} : \cdots : q_{n-k+1,\ldots,n}) \in \mathbb{P}^{\binom{n}{k}-1}$ such that all coordinates except for $q_{I \cup \{i\}}, q_{I \cup \{j\}}$ are zero, and $q_{I \cup \{i\}} = |p_{I \cup \{j\}}|$ and $q_{I \cup \{j\}} = |p_{I \cup \{i\}}|$. Then, all Plücker relations are satisfied so this point represents an element $A \in \mathrm{Gr}_{\geq 0}(k, n)$. We have $(\wedge_k A)(\wedge_k(Z^T M)) = 0$, so the hyperplane given by $M$ intersects the Grasstope $\mathcal{G}_{k,m,n}(Z)$. $\square$

Note that when $m = 1$, every hyperplane in $\mathbb{P}^{\binom{k+1}{k}-1}$ corresponds to some point in $\mathrm{Gr}(k, k + 1)$. Then by choosing a hyperplane disjoint from $\mathcal{G}_{k,1,n}(Z)$ to be the hyperplane at infinity, we arrive at the following result.

**Corollary 2.1.12.** *The Grasstope $\mathcal{G}_{k,1,n}(Z)$ is tame if and only if its image under the Plücker embedding is contained in some affine chart of $\mathbb{P}^{\binom{k+m}{k}-1}$.*

Tame Grasstopes share many nice properties with amplituhedra. In particular, for $m = 1$ they are homeomorphic to closed balls and can be described as complexes of bounded cells of affine hyperplane arrangements [KW19, Section 9]. The focus of this chapter, however, is to study more general Grasstopes, which may behave somewhat less regularly.

**Definition 2.1.13** (Wild Grasstope). If the map $\tilde{Z}$ is well-defined on $\mathrm{Gr}(k,n)_{\geq 0}$ but $Z$ does not satisfy (2.2), then the Grasstope $\mathcal{G}_{k,m,n}(Z)$ is called *wild*.

## 2.2 Results

We first extend [KW19, Corollary 3.19] from the $\mathcal{B}$-amplituhedron to primal Grasstopes.

**Theorem 2.2.1.** *Let $Z$ be a real $(k+1) \times n$ matrix such that $\hat{Z}$ has no base locus inside of $\mathrm{Gr}_{\geq 0}(k,n)$. Let $W$ be the rowspan of $Z$. Then*

$$\mathcal{H}_{k,1,n}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) \geq k\}.$$

*Proof.* For one inclusion, suppose that $w = V^\perp \cap W$ for some $V \in \mathrm{Gr}_{\geq 0}(k,n)$. Then by Theorem 2.1.9, we must have that $\overline{\mathrm{var}}(w) \geq k$.

For the other inclusion, suppose that $\overline{\mathrm{var}}(w) \geq k$. Then $\mathrm{var}(\mathrm{alt}(w)) \leq n - k - 1$, so we can extend $\mathrm{alt}(w)$ to some $V \in \mathrm{Gr}^{\geq 0}(k,n)$. We may write $V = \mathrm{alt}(V^\perp)$ by [KW19, Lemma 3.3(ii)]. Then $w$ is contained in $V^\perp$. $\qquad\qquad\square$

For a rational Grasstope, we still have that any $w$ with $\overline{\mathrm{var}}(w) \geq k$ lies in some $V^\perp$ with $V$ in the nonnegative Grassmannian. The issue is that $V^\perp \cap W$ may have dimension greater than one, so that $V$ lies in the base locus of the map. However, we can still characterize the *open Grasstope* in terms of sign patterns.

**Theorem 2.2.2.** *Let $Z$ be a real full-rank $(k+1) \times n$ matrix with rowspan $W$. Then*

$$\mathcal{H}^\circ_{k,1,n}(W) = \{w \in \mathbb{P}(W) : \mathrm{var}(w) \geq k\}.$$

*Proof.* For one inclusion, suppose that $w = V^\perp \cap W$ for some $V \in \mathrm{Gr}_{>0}(k,n)$. Then by Theorem 2.1.9, we must have that $\mathrm{var}(w) \geq k$.

For the other inclusion, suppose that $\mathrm{var}(w) \geq k$. Let $\mathcal{V}$ denote the Schubert variety of $(n-k)$-spaces containing $w$. Then the intersection of $\mathcal{V}$ with $\mathcal{U} := \{V^\perp : V \in \mathrm{Gr}_{>0}(k,n)\}$ is non-empty, by the same argument as Theorem 2.2.1. The map which sends $V \in \mathrm{Gr}(k,W)$ to $V^\perp \in \mathrm{Gr}(m,W)$ is open (in the Euclidean topology), and so is the positive Grassmannian. Therefore $\mathcal{U}$ is open. Thus $\mathcal{V} \cap \mathcal{U}$ is open and nonempty in $\mathcal{V}$.

Consider the Schubert variety $\mathcal{W}$ of $(n-k)$-spaces meeting $W$ in dimension at least two. In other words, $\mathcal{W}$ is the base locus of the map $\hat{Z}$. It remains to show that $\mathcal{W}$ intersects $\mathcal{V}$ in

a proper subvariety. But this is true; indeed, $\langle w \rangle \oplus W^\perp$ has dimension $n-k$ and intersects $W$ only in $w$, so it is a point in $\mathcal{V} \setminus \mathcal{W}$. Now, we may find a vector space $V \in (\mathcal{V} \cap \mathcal{U}) \setminus (\mathcal{V} \cap \mathcal{W})$, so that $V^\perp$ lies in the positive Grassmannian but outside the base locus. $\qquad\square$

Throughout this section, $W$ will denote the rowspan of $Z$, as in Theorem 2.2.2. In order to use these theorems to obtain a description of the Grasstope $\mathcal{G}_{k,1,n}(Z)$ in terms of coordinates, we must describe the map $\mathrm{Gr}(k, k+1) \to \mathrm{Gr}(1, W)$ explicitly. We briefly review the definition of twistor coordinates from Section 1.1.

**Definition 2.2.3** (Twistor coordinates). Let $Z$ be a real $(k+m) \times n$ matrix of full rank. Let $Y$ be a $k \times k + m$ matrix of full rank, whose rowspan $[Y]$ is a point in $\mathrm{Gr}(k, k+m)$. The *twistor coordinates* of $[Y]$ with respect to $Z$ are defined as

$$\langle Y\, i_1 \ldots, i_m \rangle := \det \begin{bmatrix} | & & | & \\ Z_{i_1} & \ldots & Z_{i_m} & Y^T \\ | & & | & \end{bmatrix}, \tag{2.3}$$

for each ordered subset $(i_1, \ldots, i_m)$ of $[n]$.

We now focus on the case $m = 1$, in which case these coordinates are a convenient way to describe the map from $\mathrm{Gr}(k, k+1)$ to $\mathrm{Gr}(1, W)$. Let $Z$ be a real $n \times (k+1)$ matrix of full rank. The space $\mathrm{Gr}(k, k+1)$ is isomorphic to dual projective space $(\mathbb{P})^k$, which parameterizes $k$-dimensional vector subspaces of $\mathbb{R}^{k+1}$. Let $\mathbf{x}$ be a vector of length $k+1$, whose entries are the Plücker coordinates of a point $[\mathbf{x}]$ in $\mathrm{Gr}(k, k+1)$. Then the twistor coordinates of $[\mathbf{x}]$ are linear forms in the entries of $\mathbf{x}$. We relabel these linear forms $l_1(\mathbf{x}), \ldots, l_n(\mathbf{x})$ to emphasize the dependence on $\mathbf{x}$.

**Lemma 2.2.4.** *Let $W \subset \mathbb{R}^n$ be a $(k+1)$-dimensional vector space. Let $f_Z : \mathrm{Gr}(1, W) \to \mathrm{Gr}(k, k+1)$ be the composition of $V \mapsto V^\perp \cap W$ and the isomorphism $\mathrm{Gr}(k, W) \to \mathrm{Gr}(k, k+1)$. Then in coordinates, the inverse $f_Z^{-1}$ is given by*

$$\mathbf{x} \mapsto [l_1(\mathbf{x}) : \ldots : l_n(\mathbf{x})],$$

*where the latter is a point in $\mathbb{P}^{n-1}$ lying in $\mathbb{P}W$.*

*Proof.* To see this concretely, we may rewrite the linear form $\langle \mathbf{x}\, i \rangle$ as the dot product $Z_i \cdot \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a vector of length $k+1$ whose entries are linear in the coordinates of $\mathbf{x}$. Then $\hat{\mathbf{x}} \in \mathbb{P}^k$ is the vector of coordinates representing the orthogonal complement of the vector space $[\mathbf{x}]$ in $\mathbb{R}^{k+1}$. Indeed by the determinantal expression in (2.3), $Z_i$ is in $[\mathbf{x}]$ if and only if $\langle \mathbf{x}\, i \rangle = Z_i \cdot \hat{x}$ vanishes. Concretely, we have

$$\hat{\mathbf{x}} = ((-1)^{k-1} x_k, \ldots, -x_2, x_1).$$

The hyperplane $[\mathbf{x}]$ in $\mathbb{R}^{k+m}$ gives a hyperplane in $W$ via the isomorphism $\mathrm{Gr}(k, k+m) \to \mathrm{Gr}(k, W)$. The length $n$ vector $\hat{\mathbf{x}}W$ spans the orthogonal complement in $W$ of this hyperplane. $\qquad\square$

This analysis gives the following concrete description of $m = 1$ Grasstopes in Plücker coordinates, originally given in [MPP25]. Here $L(\mathbf{x})$ is any lift to $\mathbb{R}^n$ of the point $[l_1(\mathbf{x}) : \ldots : l_n(\mathbf{x})]$; indeed, all lifts have the same sign variation vectors. Lemma 2.2.4 allows us to plug in $w = L(\mathbf{x})$ in Theorems 2.2.1 and 2.2.2, from which we deduce the following corollaries.

**Corollary 2.2.5.** *[MPP25, Theorem 3.1] Suppose $\tilde{Z} : \mathrm{Gr}(k, n) \dashrightarrow \mathrm{Gr}(k, k+1)$ is well-defined on $\mathrm{Gr}_{\geq 0}(k, n)$. Then the Grasstope $\mathcal{G}_{n,k,1}(Z)$ consists of the points $\mathbf{x} \in \mathbb{P}^k$ such that $\overline{\mathrm{var}}(L(\mathbf{x})) \geq k$, where $L(\mathbf{x})$ is the vector of twistor coordinates of $\mathbf{x}$ with respect to $Z$.*

**Corollary 2.2.6.** *[MPP25, Proposition 3.3] For a map $\tilde{Z} : \mathrm{Gr}_{\geqslant 0}(k, n) \dashrightarrow \mathrm{Gr}(k, k+1)$ given by a matrix $Z$, the open Grasstope $\tilde{Z}(\mathrm{Gr}_{>0}(k, n))$ consists of the points $\mathbf{x} \in \mathbb{P}^k$ such that $\mathrm{var}(L(\mathbf{x})) \geq k$, where $L(\mathbf{x})$ is the vector of twistor coordinates of $\mathbf{x}$ with respect to $Z$.*

We conclude this section by showing that open rational Grasstopes are indeed open.

**Proposition 2.2.7.** *The open rational Grasstope of $Z$ is open and, if $Z$ has no zero columns, the closure of the open rational Grasstope of $Z$ contains the rational Grasstope of $Z$.*

*Proof.* Let $Z$ define a rational Grasstope and consider a point $\mathbf{x}$ with $\mathrm{var}(L(\mathbf{x})) \geq k$, where $L(\mathbf{x})$ is the vector of twistor coordinates of $\mathbf{x}$ with respect to $Z$. Then for all points $\mathbf{x}'$ in a sufficiently small neighborhood around it, $L(\mathbf{x}')$ has the same signs as $L(\mathbf{x})$ in all the indices of the nonzero entries of $L(\mathbf{x})$. Since changing the zero entries cannot decrease the sign variation, the neighborhood is contained in the open rational Grasstope.

It follows from Theorem 2.1.9 that the rational Grasstope of $Z$ must be contained in the set $C := \{\mathbf{x} : \overline{\mathrm{var}}(L(\mathbf{x})) \geq k\}$. We show that when $Z$ has no zero columns, $C$ is the closure of the open rational Grasstope of $Z$. First, we show that the complement of $C$, the set $\{\mathbf{x} : \overline{\mathrm{var}}(L(\mathbf{x})) < k\}$, is open. Let $\mathbf{x}$ be such that $\overline{\mathrm{var}}(L(\mathbf{x})) < k$. Then for all points $\mathbf{x}'$ in a sufficiently small neighborhood around $\mathbf{x}$, $L(\mathbf{x}')$ has the same signs as $L(\mathbf{x})$ in all the indices of the nonzero entries of $L(\mathbf{x})$. Since changing the values of the zero entries cannot increase $\overline{\mathrm{var}}$, we know $\overline{\mathrm{var}}(L(\mathbf{x}')) < k$. Therefore $C$ is closed.

Now consider a point $\mathbf{x} \in C$ and an open neighborhood $N$ of points around it. We will show that there is some $\mathbf{x}' \in N$ with $\mathrm{var}(L(\mathbf{x}')) \geq k$, which is sufficient to conclude that $C$ is the closure of the open rational Grasstope of $Z$. Since $Z$ has no zero columns, each zero entry of $L(\mathbf{x})$ corresponds to containment of $\mathbf{x}$ in a hyperplane. Any open neighborhood around $\mathbf{x}$ contains points on either side of the hyperplane.

Similarly, if $\mathbf{x}$ lies in the intersection of several hyperplanes, any open neighborhood of $\mathbf{x}$ contains points in each orthant defined by these hyperplanes, so any sign pattern can be achieved using the zero entries in $L(\mathbf{x})$. In particular, there exists a point $\mathbf{x}'$ with signs in the nonzero entries of $L(\mathbf{x})$ equal to the signs of the corresponding entries of $L(\mathbf{x}')$, and signs of the zero entries replaced by the signs ensuring $\mathrm{var}(L(\mathbf{x}')) \geq k$. Indeed, we can restrict $N$ to be small, so that the signs in the nonzero entries of $L(\mathbf{x})$ are unchanged. $\square$

In Section 2.3, we give some examples where the rational Grasstope is equal to the closure of the open rational Grasstope. However, we do not know if this holds in general. It would be interesting to describe which boundary components are contained in $m = 1$ rational Grasstopes.

## 2.3 Examples

In this section we provide examples of the families of Grasstopes we considered. We begin with an example of a tame Grasstope.

**Example 2.3.1** (A tame Grasstope). Let

$$Z = \begin{bmatrix} 1 & 0 & -1 & -3 & -2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -2 \end{bmatrix}.$$

This matrix is not totally positive, since $p_{123} = 1$ and $p_{124} = -1$. However, the first two columns of $Z$ span a totally positive line, so $Z$ satisfies (2.2) with the matrix $M$ being

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T.$$

Therefore the resulting Grasstope is tame but is not an amplituhedron. The columns of $Z$ define five linear forms:

$$l_1 = z, \quad l_2 = -y, \quad l_3 = x - y - z, \quad l_4 = -x - 2y - 3z, \quad l_5 = -2x - y - 2z.$$

Not every affine chart that we choose results in a bounded picture. For instance, if we map $(x : y : z) \mapsto (x + y : y : z)$ and dehomogenize with respect to the first coordinate, the resulting picture is unbounded. However, as predicted by Corollary 2.1.12, there are lines disjoint from the Grasstope. One of them is $\{-4x + z = 0\}$ in $\mathbb{P}^2$. By picking it to be the line at infinity (that is, by mapping $(x : y : z) \mapsto (-4x + z : y : z)$ and dehomogenizing with respect to the first coordinate), we obtain affine lines given by the linear forms

$$\tilde{l}_1 = \tilde{y}, \quad \tilde{l}_2 = -\tilde{x}, \quad \tilde{l}_3 = \frac{-1 - 4\tilde{x} - 3\tilde{y}}{4}, \quad \tilde{l}_4 = \frac{-1 - 8\tilde{x} - 13\tilde{y}}{4}, \quad \tilde{l}_5 = \frac{1 - 2\tilde{x} - 5\tilde{y}}{2}.$$

Each line has an orientation, with the positive half-space given by the points $(\tilde{x}, \tilde{y})$ for which $\tilde{l}(\tilde{x}, \tilde{y}) \geq 0$. The lines divide the affine plane into regions, each of which has a corresponding sign vector with $i^{th}$ coordinate being $+$ if $\tilde{l}_i(\tilde{x}, \tilde{y}) > 0$ for all $(\tilde{x}, \tilde{y})$ in the region, and $-$ if $\tilde{l}_i(\tilde{x}, \tilde{y}) < 0$. The Grasstope of $Z$ consists exactly of those points in the regions for which $\overline{\text{var}}(u) \geq 2$, as can be seen in Figure 2.1. $\diamond$
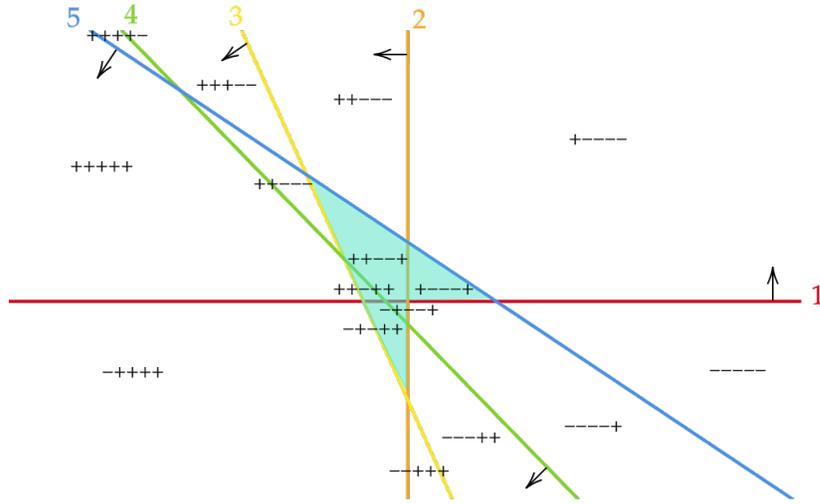
Figure 2.1: Affine chart in which the tame Grasstope is bounded. The six lines corresponding to the rows of $Z$ are colored red, orange, yellow, green, and blue, in order, with orientations given by arrows. The shaded portion of the figure is the Grasstope, which consists exactly of the regions with at least two sign changes.

**Example 2.3.2** (A wild Grasstope). Let

$$Z = \begin{bmatrix} 2 & 2 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

This example was found by Galashin and communicated to us by Lam [Lam23]. We claim that $Z$ gives a wild Grasstope. To see this, suppose there exists a $3 \times 2$ matrix $M$ such that the $6 \times 2$ matrix $Z^T M$ has positive $2 \times 2$ minors. The $2 \times 2$ minors of $Z^T M$ are the 15 entries of $\wedge_2(Z^T M) = \wedge_2(Z^T) \wedge_2 (M)$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 3 & 1 & 2 & 1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & -2 & 2 & 2 & 0 & -1 & 1 \\ -2 & -2 & -2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 3 & 1 & 0 & 2 \end{bmatrix}^T \begin{bmatrix} p_{12} \\ p_{13} \\ p_{23} \end{bmatrix},$$

where $p_{12}, p_{13}, p_{23}$ are the minors of $M$. Then, column 4 of $(\wedge_2 Z)$ tells us that $p_{12} + p_{13} > 0$ but column 10 tells us that $p_{12} + p_{13} < 0$, so no such matrix $M$ can exist.

The six columns of $Z$ correspond to the six linear forms

$$l_1 = 2x - 2y + 2z, \qquad l_2 = 2x - 3y + 2z, \qquad l_3 = -y$$
$$l_4 = -z, \qquad l_5 = 2x - 2y + z, \qquad l_6 = x.$$

Mapping $(x : y : z) \mapsto (x + y : y : z)$ and dehomogenizing with respect to the first coordinate, we obtain affine lines given by the linear polynomials

$$\tilde{l}_1 = 2\tilde{y} - 4\tilde{x} + 2, \qquad \tilde{l}_2 = 2\tilde{y} - 5\tilde{x} + 2, \qquad \tilde{l}_3 = -\tilde{x},$$
$$\tilde{l}_4 = -\tilde{y}, \qquad \tilde{l}_5 = \tilde{y} - 4\tilde{x} + 2, \qquad \tilde{l}_6 = -\tilde{x} + 1.$$

We draw these in the affine plane and color them (in order) red, orange, yellow, green, blue, and purple. We also give the lines orientations with the positive half-space given by the points $(\tilde{x}, \tilde{y})$ for which $\tilde{l}(\tilde{x}, \tilde{y}) > 0$. Then the Grasstope of $Z$ consists exactly of those points in the regions between the lines for which $\overline{\mathrm{var}}(u) \geq 2$, as can be seen in Figure 2.2.
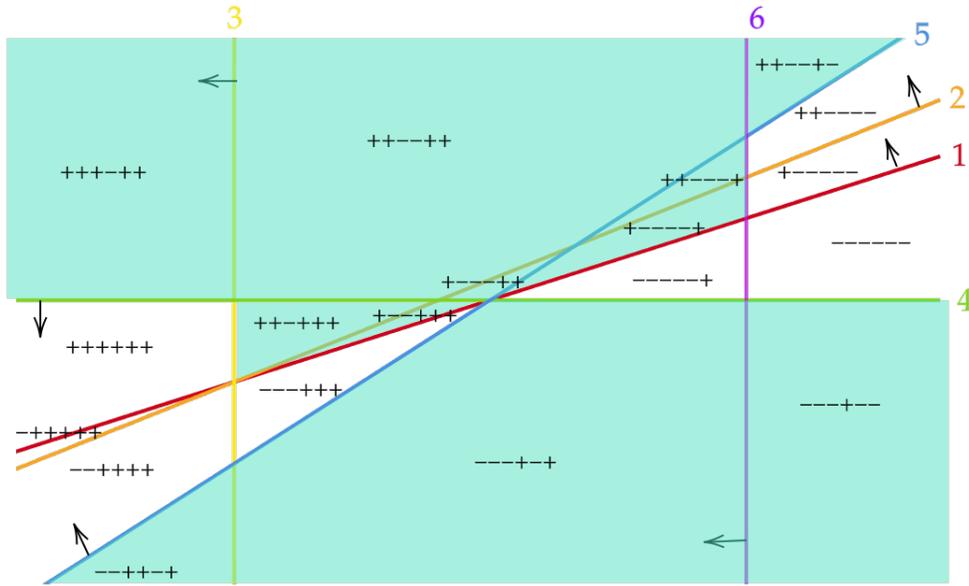


Figure 2.2: The six lines corresponding to the columns of $Z$ are pictured as described, with orientations given by the arrows. The regions can then be labelled by sign patterns. The shaded portion of the figure is the Grasstope, and it consists exactly of those regions with at least two sign changes.

$\diamond$

**Example 2.3.3** (A rational Grasstope with closed boundary and Möbius strip topology). Let

$$Z = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & 0 & -2 & -1 \end{bmatrix}.$$

Note that $(1, 1, 1, 1, 1, 1) \in \ker(Z)$, so the map $\tilde{Z}$ has base points on $\mathrm{Gr}_{\geq 0}(2, 6)$. Following Corollary 2.2.6 we can still describe its open Grasstope. As in the previous examples, we
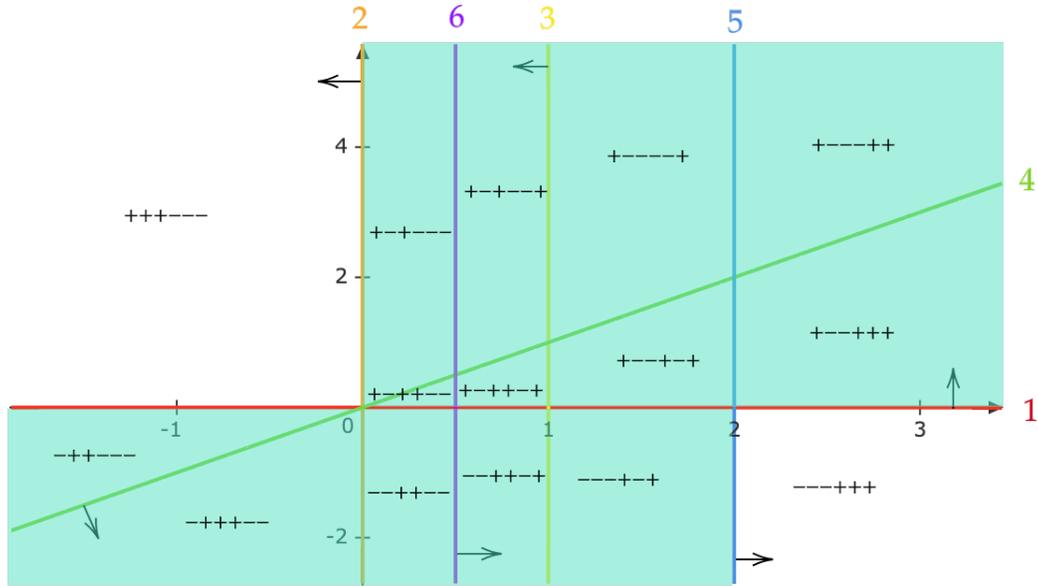
Figure 2.3: The six lines corresponding to the columns of $Z$ are pictured as described, with orientations given by the arrows. The regions can then be labelled by sign patterns. The shaded portion of the figure is the Grasstope, and it consists exactly of those regions with at least two sign changes. In this case, the shaded region is a Möbius strip.

find six dehomogenized linear forms corresponding to six affine lines

$$
\begin{aligned}
&\tilde{l}_1 = \tilde{y}, &\qquad &\tilde{l}_2 = -\tilde{x}, &\qquad &\tilde{l}_3 = -3\tilde{x} + 3, \\
&\tilde{l}_4 = \tilde{x} - \tilde{y}, &\qquad &\tilde{l}_5 = \tilde{x} - 2, &\qquad &\tilde{l}_6 = 2\tilde{x} - 1.
\end{aligned}
$$

Then we can find the open rational Grasstope of $Z$ as in Figure 2.3.

We claim that the rational Grasstope of $Z$ is the closure of the open rational Grasstope. One can check this by directly finding Plücker coordinates for some $M \in \mathrm{Gr}_{\geq 0}(2,6)$ which map to the points that lie on the boundary. For example, to find such $M$ for any point of the form $(0, a)$ with $a \geq 0$, we solve $\wedge_2(Z^T) \wedge_2 (M) = (1 : 0 : a)$. Then $\wedge_2(M)$ must have nonnegative entries which satisfy the Plücker relations. This process is made easier if one recalls that the Plücker relations are trivially satisfied if all entries are zero except for ones which correspond to pairwise overlapping submatrices, that is any two nonzero minors come from submatrices which share a column. In this case, since

$$
\wedge_2(Z) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 3 & -2 & 0 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & -2 & -3 & 2 & 0 & -1 & 3 & 1 & -3 \end{bmatrix},
$$

we find that the point $(1 : 0 : a)$ is given by $\wedge_2(M) = (0 : 0 : a/3 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$. Since the only nonzero entries correspond to the minors $p_{23}$ and $p_{24}$, the Plücker

relations are satisfied. Thus the portion of the boundary line $x = 0$ with $y \geq 0$ is part of the rational Grasstope of $Z$. One can similarly check that all other parts of all six lines are included. Therefore the rational Grasstope of $Z$ is closed. Furthermore, topologically, as one can see from Figure 2.3, it is a Möbius strip. $\diamond$

We conclude this section by remarking that rational Grasstopes may not be simply connected. Grasstopes where the map $\tilde{Z}$ is well-defined will be simply connected; indeed, any loop in $\mathcal{H}^\circ_{k,m,n}(Z)$ can be lifted to a loop in the nonnegative Grassmannian, which is contractible because the former is homeomorphic to an closed ball. However, the topological space $\mathrm{Gr}_{\geq 0}(k,n) \setminus (\Omega(\tilde{Z}) \cap \mathrm{Gr}_{\geq 0}(k,n))$ may not be homemorphic to a closed ball [GKL22]. For example, the map given by

$$Z = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

has as its base locus the point $[1 : 1 : 1]$, which lies in $\mathbb{P}^2_{\geq 0}$. Its complement in nonnegative projective space is homeomorphic to $S^1$. The following gives a concrete example of a Grasstope with a non-contractible loop.

**Example 2.3.4.** Let $Z$ be any $3 \times 5$ matrix with positive maximal minors, and multiply columns $1, 3$ and $4$ each by $-1$. By the sign characterization of Corollary 2.2.6, the complement of the Grasstope consists of the pink-shaded regions in Figure 2.4. $\diamond$
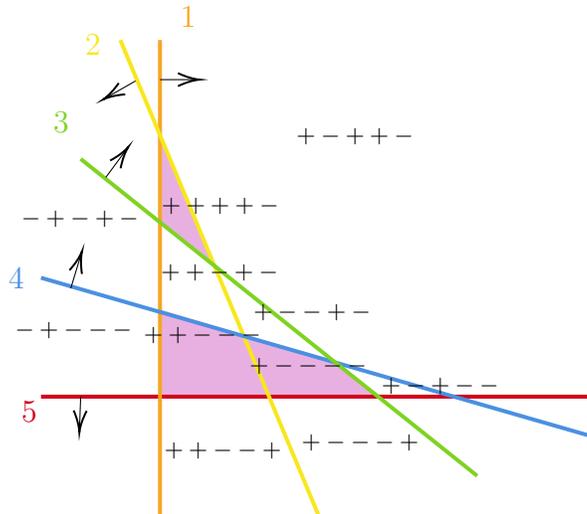


Figure 2.4: The (closed) pink-shaded regions form the complement of a rational Grasstope

## 2.4 Background on oriented matroids

Much of the material on hyperplane arrangements can be naturally generalized to oriented matroids [Bjö+99]. In this section, we review the basics of oriented matroid theory and

recall a dictionary between hyperplane arrangements and oriented matroids. This will prove useful in algorithmically counting the number of regions in a given Grasstope (see Section 2.5). We begin by discussing signed circuits, which we will often abbreviate as circuits if the meaning is clear from context.

**Definition 2.4.1** (Signed circuit axioms). An oriented matroid consists of a ground set $\mathcal{E}$ and a collection $\mathcal{C}$ of tuples of the form $X = (X^+, X^-)$ called *circuits*, where $X^+, X^-$ are disjoint subsets of $\mathcal{E}$ satisfying

1. $\emptyset$ is not a circuit.

2. If $X$ is a circuit, then so is $-X = (X^-, X^+)$.

3. No proper subset of a circuit is a circuit.

4. (Elimination). If $X_0$ and $X_1$ are two signed circuits with $X_0 \neq X_1$ and $e \in X_0^+ \cap X_1^-$, then there is a third circuit $X \in \mathcal{C}$ with $X^+ \subset (X_0^+ \cup X_1^+) \setminus \{e\}$ and $X^- \subset (X_0^- \cup X_1^-) \setminus \{e\}$.

We can obtain the oriented matroid of a matrix as follows.

**Definition 2.4.2** (Oriented matroid of a matrix). Fix a matrix $A$ and let $\sum_i \lambda_i v_i$ be a minimal linear dependency among its columns. Associate to this dependency the signed set $X = (X^-, X^+)$, where

$$X^- = \{i : \lambda_i < 0\}$$
$$X^+ = \{i : \lambda_i > 0\}.$$

Then the *oriented matroid* $\mathcal{M}_A$ associated to $A$ has as its signed circuits the signed sets coming from minimal linear dependencies.

One can check that oriented matroids of matrices satisfy the signed circuit axioms.

**Example 2.4.3** (Matroid of a matrix $A$). Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Then the only linear dependency up to scaling is $v_4 - 4v_3 + 3v_2 - v_1 = 0$. Thus the only circuit (when only one of $X, -X$ is considered) is $24\overline{13}$, where we use a more compact notation in which the bar indicates being in the negative part of the signed set. $\diamond$

**Remark 2.4.4.** One can also define $\mathcal{M}_A$ by signed bases; the matroid is the map which assigns to each size $k$ subset $I \subset [n]$ the sign of the determinant of $A_I$. This definition is called the *chirotope definition* [Bjö+99, page 6] and satisfies the *chirotope axioms,* which we do not describe here. In particular, the database [Fin] used in Section 2.5 indexes matroids by chirotope.

We recall a few definitions we will need to explain how a hyperplane arrangement can be viewed as an oriented matroid.

**Definition 2.4.5** (Composition). Let $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ be signed sets. Then their *composition* $X \circ Y$ is $(X^+ \cup (Y^+ \setminus X^-), X^- \cup (Y^- \setminus X^+))$.

**Definition 2.4.6** (Orthogonality). Let $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ be signed sets. Define $S(X, Y) = (X^+ \cap Y^-) \cup (X^- \cap Y^+)$. We say $X$ and $Y$ are *orthogonal* if $S(X, Y)$ and $S(X, -Y)$ are both empty or both non-empty.

We define *vectors* as compositions of circuits, and *cocircuits* and *covectors* as the circuits and vectors of the dual matroid [Bjö+99, page 4], respectively. An equivalent definition which is easier for computing is that the covectors of $\mathcal{M}$ are the signed sets which are orthogonal to all vectors of $\mathcal{M}$. For more detail, see [Bjö+99, Chapter 1]. There is yet another definition in the case of vector configurations.

**Definition 2.4.7** (Oriented matroid of a vector configuration). One can also view the columns $v_i$ of the matrix $A$ as vectors in $\mathbb{R}^k$. For such a vector configuration, the covectors can be defined as the set of tuples $Y_H = (Y_H^+, Y_H^-)$ as $H$ runs over oriented hyperplanes, where $Y_H^+$ is the set of vectors in the positive halfspace defined by $H$, and $Y_H^-$ is the set of vectors in the negative halfspace. The *cocircuits* are the minimal covectors. They arise from hyperplanes that are spanned by subsets of $\{v_1, ..., v_n\}$.

**Definition 2.4.8** (Oriented matroid of a hyperplane arrangement). Let $H_i$ be the hyperplane given by the vanishing of $l_i(x) = a_i \cdot x$. Then the oriented matroid of the arrangement $\{H_i\}_{i=1}^n$ is the matroid of the vector configuration given by $a_1, ..., a_n$, or equivalently, the oriented matroid of the matrix with columns $a_1, ..., a_n$.

**Remark 2.4.9.** Faces of a hyperplane arrangement correspond to covectors of its oriented matroid. Regions correspond to maximal covectors. The *rank* (denoted by $r$) of the oriented matroid is $k + 1$, where $k$ is the dimension of the ambient space [Bjö+99, Chapter 1].

**Example 2.4.10** (Matroid of a matrix $A$). As in Example 2.4.3, let $A$ be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The cocircuits are obtained considering hyperplanes $H$ spanned by pairs of rows. For example, if $H = \text{Span}\{a_1, a_3\} = \{y = 0\}$, then $a_2$ is in the positive halfspace (since the $y$-coordinate is 1) and $a_4$ is in the negative halfspace (since the $y$ coordinate is $-3$). Thus we obtain the cocircuit $2\bar{4}$.

The total set of cocircuits (not including negations) is $\{2\bar{4}, 34, 23, 14, 3\bar{1}, 12\}$. In the previous sections we have used sign vector notation; that is, to each signed set we associate a length $n$ vector with $\pm$ at index $i$ if $i$ is in $X^{\pm}$ and 0 otherwise. Applying this convention,

one can check that the covectors are exactly the sign vectors with fewer than 3 sign changes. For example, $2\overline{4} \circ 3\overline{1} = 23\overline{14}$, which corresponds to $(-+ +-)$. ◇

**Definition 2.4.11.** A matroid is *realizable* if it arises from a real hyperplane arrangement.

## 2.5 Oriented matroid Grasstopes

Given an $(k + 1) \times n$ matrix $Z$, one may ask questions about the topology of its $m = 1$ Grasstope. For instance, is it closed, connected, contractible? How many regions of the hyperplane arrangement does it contain? For the $m = 1$ amplituhedron, the answer to the first three questions is positive [KW19, Corollary 6.18]. The $m = 1$ amplituhedron contains as few regions as possible for a simple hyperplane arrangement; that is, all sign vectors with variation less than $k$ appear as labels of regions in the arrangement [KW19, Proposition 6.14]. In this section, we investigate this question for more general $m = 1$ Grasstopes.

**Definition 2.5.1.** A *region* of a hyperplane arrangement is a connected component of the complement of the union of hyperplanes in the arrangement.

**Definition 2.5.2.** Let $\mathcal{A}$ be an arrangement of $n$ hyperplanes in $\mathbb{A}^k$. Then $\mathcal{A}$ is called *simple* if the intersection of any $i$ hyperplanes in $\mathcal{A}$ has codimension $i$ in $\mathbb{A}^k$ for all $i \leq k$, and is empty for $i > k$.

In [Zas75], Zaslavsky gives the following formulae for the number of total regions $t(\mathcal{A})$ and bounded regions $b(\mathcal{A})$ of a simple affine arrangement $\mathcal{A}$ of $n$ hyperplanes in $\mathbb{A}^k$:

$$t(\mathcal{A}) = 1 + n + \binom{n}{2} + \ldots + \binom{n}{k},$$
$$b(\mathcal{A}) = \binom{n-1}{k}.$$

The former count first appeared in work of Buck [Buc43]. Note that since intersections of two hyperplanes in any projective arrangement $\mathcal{P}$ have codimension two in $\mathbb{P}^k$ and there are a finite number of them, we can always find some hyperplane avoiding them, and hence an affine chart which contains all of them (see Figure 2.5). A projective arrangement $\mathcal{P}$ naturally induces an affine arrangement $\mathcal{A}$ in this affine chart. We call $\mathcal{P}$ *simple* if this corresponding affine arrangement is simple. If a region of $\mathcal{P}$ intersects the chosen hyperplane at infinity, it induces two unbounded regions of $\mathcal{A}$. Otherwise it induces a single bounded region. Thus, the total number of regions in a simple arrangement $\mathcal{P}$ is

$$r(\mathcal{P}) = b(\mathcal{A}) + \frac{t(\mathcal{A}) - b(\mathcal{A})}{2}. \tag{2.4}$$

From here on we assume our arrangements are simple. We write $\beta(k, n)$ for the number of possible sign patterns of length $n$ with sign variation less than $k$ and $\gamma(k, n)$ for the number
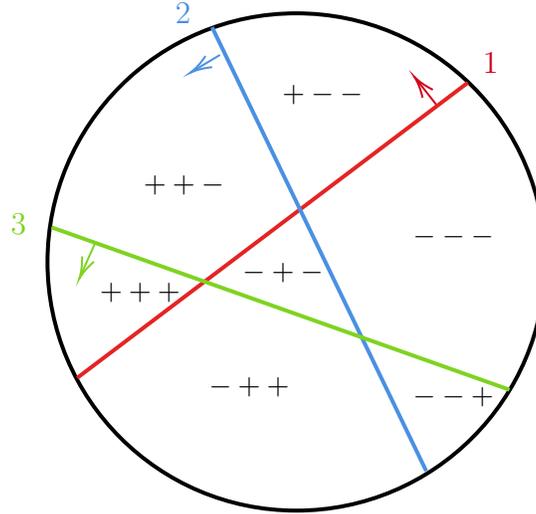
Figure 2.5: An oriented hyperplane arrangement and its sign labels.

of sign patterns with variation greater or equal than $k$ (we identify sign patterns $\sigma$ and $-\sigma$). Note that $\beta(k,n) = 1 + (n-1) + \binom{n-1}{2} + ... + \binom{n-1}{k}$, and $\gamma(k,n) = 2^{n-1} - \beta(k,n)$. Corollary 2.2.5 and Proposition 2.2.7 then give the lower bound $r(\mathcal{P}) - \beta(k,n)$ for the number of regions in $\mathcal{G}_{k,1,n}(Z)$, where $\mathcal{P}$ is the hyperplane arrangement defined by $Z$. An upper bound is given by the minimum of $\gamma(k,n)$ and $r(\mathcal{P})$.

The database [Fin] contains a catalog of isomorphism classes of oriented matroids [Fin01, Section 6]. Each matroid is indexed by a vector of signs of its bases. Recall from the previous section that any central arrangement produces an oriented matroid. Each projective arrangement $\mathcal{P} \subset \mathbb{P}^k$ produces an oriented matroid: we take the cone over $\mathcal{P}$ to get a central arrangement in $\mathbb{A}^{k+1}$. Observe that an arrangement is simple if and only if the underlying (non-oriented) matroid is uniform.

We iterate over all uniform oriented matroid isomorphism classes in this catalog for small values of $k$ and $n$. Note that, for all the values of $k$ and $n$ which we consider, all uniform matroids are realizable [FMM13], that is, arise from hyperplane arrangements. Within each isomorphism class, we iterate over all possible reorderings of the ground set and reorientations. If the matroid is realizable, at the level of the matrix $Z$ defining the arrangement as described in Section 2.1, reorderings correspond to permuting the rows and reorientations to negating certain rows. For each isomorphism class, ordering of the hyperplanes, and a choice of orientation, we compute the number of regions in the corresponding Grasstope (that is, the number of maximal covectors with sign variation greater or equal than $k$; see Remark 2.4.9). It turns out that for many values of $k$ and $n$ the minimal and maximal number of regions in the Grasstope when iterating over reorderings and reorientations does not depend on the oriented matroid isomorphism class. The minimal and maximal number of regions in the Grasstope for these values of $k$ and $n$ are presented in Table 2.1. The `Python` code

used to extract this data is available at https://mathrepo.mis.mpg.de/Grasstopes. We therefore have a computational proof of the following statement.

**Proposition 2.5.3.** *For each pair of values of $k$ and $n$ in Table 2.1 the minimal and maximal possible number of regions in a Grasstope arising from a simple arrangement of $n$ hyperplanes in $\mathbb{P}^k$ do not depend on the choice of arrangement.*

Out of the entries in Table 2.1, note that for $k = 2$ and the pairs $(3, 5)$ and $(4, 6)$, there is only one (uniform) oriented matroid up to isomorphism [Fin]. For $k = 2$, this is because any matrix may be turned into a totally positive matrix by permuting the rows. This can be done by viewing rows as vectors in the plane and arranging them in counterclockwise position.

| $k$, $n$ | Minimal | Maximal | $r(\mathcal{P})$ | $\beta(k, n)$ | $\gamma(k, n)$ |
|---|---|---|---|---|---|
| 2, 6 | 10 | 16 | 16 | 6 | 26 |
| 2, 7 | 15 | 22 | 22 | 7 | 57 |
| 3, 5 | 4 | 5 | 15 | 11 | 5 |
| 3, 6 | 10 | 16 | 26 | 16 | 16 |
| 4, 6 | 5 | 6 | 31 | 26 | 6 |
| 4, 7 | 15 | 22 | 57 | 42 | 22 |

Table 2.1: Minimal and maximal possible number of regions in a Grasstope.

The pair $(3, 7)$ does not appear in Table 2.1. This is the first time we see variation depending on which simple arrangement we choose, with the maximal number of regions ranging from 38 to 42. The reorientations and reorderings of a totally positive matrix (i.e. the amplituhedron case) give at most 42 regions, while all other oriented matroid classes achieve fewer. We can see the maximal numbers of regions for other small $k, n$ in Table 2.2.

| $k$, $n$ | Maximal | $r(\mathcal{P})$ | $\gamma(k, n)$ |
|---|---|---|---|
| 3, 7 | 42 | 42 | 42 |
| 3, 8 | 64 | 64 | 99 |
| 4, 8 | 64 | 99 | 64 |
| 5, 8 | 29 | 120 | 29 |
| 2, 9 | 37 | 37 | 247 |
| 3, 9 | 93 | 93 | 219 |
| 4, 9 | 163 | 163 | 163 |

Table 2.2: Maximal number of regions from reorienting and reordering a positive matrix.

**Example 2.5.4.** Any totally positive $3 \times 6$ matrix with the second and fourth rows negated yields a Grasstope which includes all 16 regions counted by Equation (2.4). The resulting hyperplane arrangement is cyclic, with just two orientations flipped. See Figure 2.6 to see all of the regions labelled with sign patterns.
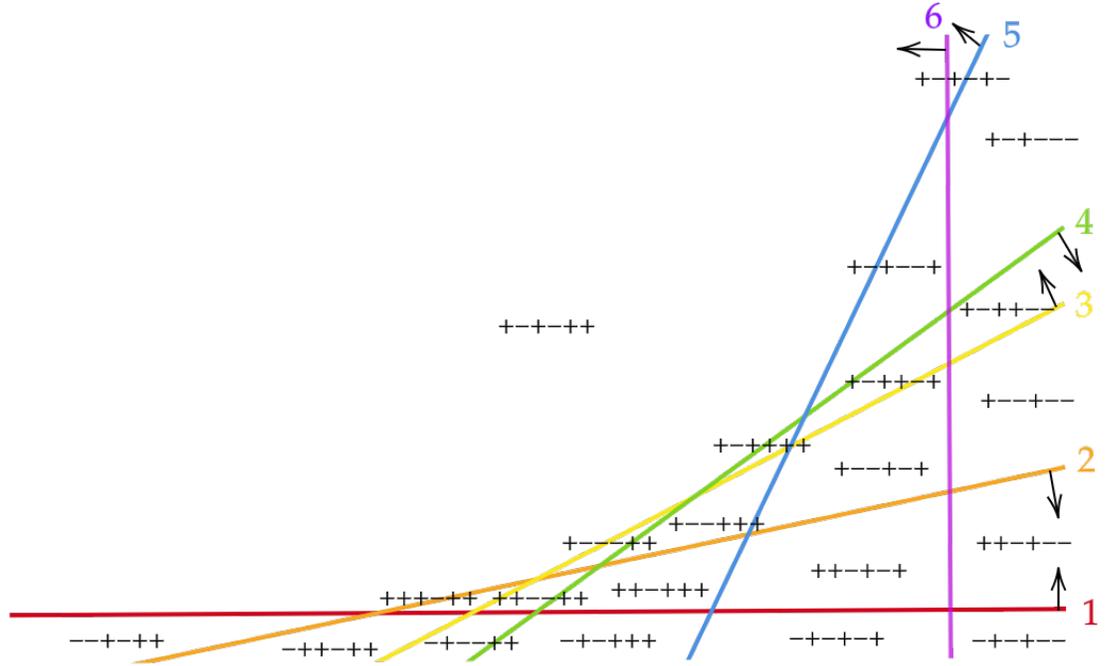


Figure 2.6: The Grasstope of a totally positive matrix with two columns negated. The six lines are cyclically ordered with orientations indicated by arrows. Every region has at least two sign changes, so the Grasstope is all of $\mathbb{P}^2$.

$\diamond$

**Example 2.5.5.** An example of a $3 \times 6$ matrix whose Grasstope has 16 regions is any totally positive matrix with the 2nd and 4th rows swapped. For examples of totally positive matrices, one can take the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & d_6^2 \end{bmatrix}.$$

with $0 < d_1 < \cdots < d_6$. $\diamond$

Note that the lower bound $r(\mathcal{P}) - \beta(k, n)$ for the number of regions in a Grasstope is attained for all $k$ and $n$, by any tame $m = 1$ Grasstope whose arrangement is simple [KW19, Proposition 6.14, Theorem 6.16]. In particular, it is attained by the $m = 1$ amplituhedron.

The upper bound is also attained by reorientation class of the $m = 1$ amplituhedron for the values in Tables 2.1 and 2.2. One interesting question to study is to determine whether this holds in general, and to describe all oriented matroids achieving this upper bound.

**Problem 2.5.6.** For any oriented matroid $\mathcal{M}$, Let $f(\mathcal{M})$ be the maximum over the reorientations of $\mathcal{M}$ of #{covectors with sign variation greater than or equal to $k$}. Let $\mathcal{M}_{pos}$ be the oriented matroid of a totally positive matrix. Is $f(\mathcal{M}_{pos})$ equal to the minimum of $\gamma(k, n)$ and $r(\mathcal{P})$?

**Problem 2.5.7.** Which oriented matroids achieve the upper bound in Problem 2.5.6?

The dictionary between hyperplane arrangements and oriented matroids (Remark 2.4.9) guides us to generalize the definition of an $m = 1$ Grasstope to oriented matroids that are not necessarily realizable, such that the definitions agree when $\mathcal{M} = \mathcal{M}_Z$.

**Definition 2.5.8** (Grasstope of an oriented matroid)**.** Let $\mathcal{M}$ be an oriented matroid of rank $r$, and $<$ be a total order on the ground set $\mathcal{E}$ of $\mathcal{M}$. Then we define the *Grasstope* $\mathcal{G}(\mathcal{M}, <)$ to be the subset of covectors $\{v : \overline{\mathrm{var}}(v) \geq r - 1\}$, where $r$ is the rank of $\mathcal{M}$ and the variation is with respect to $<$.

Note that by Topological Representation Theorem [FL78, Theorem 20], every oriented matroid arises from a pseudohyperplane arrangement, with covectors labelling the cells of this arrangement. In particular, the Grasstope $\mathcal{G}(\mathcal{M}, <)$ can be identified with the union of cells of a pseudohyperplane arrangement that satisfy the sign variation condition from Definition 2.5.8. Therefore, Grasstopes of oriented matroids are meaningful geometric objects, and topological concepts such as connectedness and contractibility generalize naturally to them. Studying their topological properties is an interesting topic for future research. Finally, we use our code to analyze a non-realizable example, which does not attain the upper bound.

**Example 2.5.9.** Consider the non-realizable matroid FMR(8) of rank 4 on 8 elements, whose signed cocircuits are given in [RS88, Table 1]. Reorientations and reorderings give at least 34 and at most 63 regions. Thus, unlike the amplituhedron, FMR(8) does not achieve the upper bound of 64. ◇

To summarize, this chapter has showcased what is special about the $m = 1$ amplituhedron as compared to other $m = 1$ Grasstopes. First of all, $m = 1$ amplituhedra are topologically very nice; while they are homeomorphic to closed balls, more general Grasstopes need not be. Fundamentally, this niceness comes from the fact that the base locus of a positive map lies outside the positive Grassmannian. In this way we see how the positivity assumption on $Z$ interacts with the positive locus in the Grassmannian to affect the topology of the projected semialgebraic set.

Moreover, we have shown that amplituhedra are extremal among $m = 1$ Grasstopes. It is a classic theorem of McMullen that cyclic polytopes, the $k = 1$ amplituhedra, maximize the number of possible faces for a fixed number of vertices and dimension [McM93]. The results

of Section 2.2 give us an analogue for $m = 1$ amplituhedra: they minimize the number of bounded regions among $m = 1$ Grasstopes with $n$ and $k$ fixed. Again, this shows us what is special about amplituhedra among Grasstopes.

Taking inspiration again from the case of polytopes, it is natural to ask when a Grasstope is a positive geometry, as in Definition 1.2.3. In the case where a Grasstope is orientable and simply connected, it is automatically a *pseudo-positive geometry* by [ABL17, Equation 3.1]. A pseudo-positive geometry is a pair $(X, X_{\geq 0})$ satisfying axioms similar to those of a positive geometry, but allowing $X_{\geq 0}$ to be empty; see [ABL17, Section 2.2] for the precise definition. However, Grasstopes are not always positive geometries. For example, the projective line $\mathbb{P}^1_{\mathbb{R}}$ is a pseudo-positive geometry, but not a positive geometry. It is the Grasstope for, for example, the matrix

$$Z = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

In many cases, including that of $m = 1$ amplituhedra, the boundary is nonempty and we do obtain positive geometries. This gives an interesting class of non-convex positive geometries.

Finally, the definition of Grasstopes of non-orientable matroids gives a fruitful avenue for future work. Since the paper [MPP25] was published, Eur and Lam have developed a notion of the canonical form of a tope of a matroid [EL26]. This generalizes the canonical form of a region of a hyperplane arrangement, which is a polytope. It would be interesting to use these forms to study $m = 1$ Grasstopes.

# Chapter 3

# Exterior Cyclic Polytopes

This chapter is based on the paper *Exterior Cyclic Polytopes and Convexity of Amplituhedra*, with Elia Mazzucchelli [MP25]. It is slightly abridged; it does not contain either of the appendices, and some material in Section 6 is removed.

The work in this chapter is motivated by the following observation: the nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$ is equal to the intersection of $\mathrm{Gr}(k, n)$ with the standard simplex in the ambient projective space. In this chapter we study whether more general amplituhedra may be described as the intersection of the Grassmannian with a convex polytope. With this motivation, we define the *exterior cyclic polytope* $C_{k,m,n}(Z)$ as the projection of the standard simplex in $\mathbb{P}^N_{\mathbb{R}} \cong \mathbb{P}(\bigwedge^k \mathbb{R}^n)$ under the linear map $\wedge^k Z : \mathbb{P}(\bigwedge^k \mathbb{R}^n) \dashrightarrow \mathbb{P}(\bigwedge^k \mathbb{R}^{k+m})$. The case $k = 1$ recovers the cyclic polytope. Our main result is that the following holds for $k = m = 2$.

**Theorem 3.5.7.** We have that

$$\mathcal{A}_{2,2,n}(Z) = \mathrm{Gr}(2, 4) \cap C_{2,2,n}(Z), \tag{3.1}$$

for every real $4 \times n$ matrix $Z$ with positive maximal minors.

More generally, we define a connected set in the Grassmannian to be *extendably convex* if it is the intersection of the Grassmannian with a convex body in the ambient Plücker space. This notion of convexity was first considered by Busemann [Bus61]. We show that the amplituhedron $\mathcal{A}_{2,2,n}(Z)$ is extendably convex for every positive $4 \times n$ matrix $Z$.

Furthermore, we uncover a relationship between the dual of the exterior cyclic polytope and the *twist* $\tau(Z)$ of the matrix $Z$. The twist map $\tau$ has been studied extensively by both mathematicians [BFZ96; GL20; MS17] and physicists [ATT18] in relation to parity duality. In the special case $k = 2$ and $m = 2$, the physical interpretation of the twist map is that it exchanges the maximal-helicity-violating (MHV) sector of the one-loop amplitude with its parity conjugate ($\overline{\mathrm{MHV}}$).

**Theorem 3.4.10.** We have that

$$\widetilde{C}_{2,2,n}(Z) = C_{2,2,n}(W)^*, \tag{3.2}$$

where $W = \tau(Z)$ is the twist map applied to $Z$, according to Definition 3.1.3.

Here the polytope $\widetilde{C}_{k,m,n}(Z)$ is obtained from the exterior cyclic polytope $C_{k,m,n}(Z)$ by keeping only those facet inequalities such that the supporting hyperplane is a Schubert divisor when restricted to $\mathrm{Gr}(k, k+m)$. We call these *Schubert facets*. We prove Proposition 3.4.10 by studying the linear matroid of $\wedge^k Z$. For the case $k = 2$, this is equal to the dual of the *hyperconnectivity matroid*, which was introduced by Kalai in [Kal85] to study highly connected graphs and has been studied more recently in the context of graph rigidity [Bra+24; CS23]. We characterize Schubert facets for $k = m = 2$.

**Theorem 3.4.5.** The Schubert facets of the polytope $C_{2,2,n}(Z)$ are exactly the $\binom{n}{2}$ hyperplanes given by the vanishing of $\langle Y \overline{ij} \rangle$ for $1 \le i < j \le n$, where $\overline{ij} := (i-1\,ii+1) \cap (j-1\,jj+1)$. Furthermore, these Schubert facets intersect transversally in $\mathrm{Gr}(2,4)$ for every $Z \in \mathrm{Mat}_{>0}(4, n)$.

The connection between the amplituhedron and physics arises in the context of positive geometries [ABL17]. The conjecture [AT14] is that the (tree) amplituhedron $\mathcal{A}_{k,4,n}(Z)$ is a positive geometry and that its canonical form computes the tree-level scattering amplitude. The case $k = m = 2$, while seemingly specific, is the only case in which this conjecture has been proven [RST24]. It coincides with the one-loop amplituhedron for $k = 0$ and $m = 4$ [ABL17, Section 6.4].

Our motivation for studying convexity and duality comes from this context of positive geometries. In many cases of physical significance the canonical function of the positive geometry has a representation as an integral over a "dual" geometry with respect to a nonnegative measure. In the prototypical example of a polytope in projective space, the canonical function can be expressed as the Laplace transform of the characteristic function on the dual polytope [ABL17, Section 7.4.2]. The existence of such a representation relies on the convexity of the polytope. The long-standing hope, going back to Hodges [Hod13], is that such a description also exists for the canonical form of the amplituhedron. This would imply that amplitudes in $\mathcal{N} = 4$ super Yang-Mills are volumes of certain geometric objects. Such an integral representation could also potentially imply that amplitudes possess a strong analytic property called complete monotonicity, which has been observed to hold for many functions appearing in quantum field theories [HR25]. In Section 3.6 we give a candidate for the underlying semialgebraic set of the dual amplituhedron. This is related to the dual of the exterior cyclic polytope, and we describe it in the case of $k = m = 2$.

**Corollary 3.6.6.** The extendable dual amplituhedron for $k = m = 2$ in twistor space is equal to the twisted amplituhedron, which has a semialgebraic description as in eq. (3.46).

The outline of the chapter is as follows. In Section 3.1 we review twistor notation, which we use to represent linear spaces in projective space as elements in the (projective) exterior algebra. We also recall the definition of the twist map on matrices with positive maximal minors, first introduced in the context of cluster algebras in [BFZ96]. In Section 3.2, we

extend Busemann's definition in [Bus61] to any semialgebraic set in a real embedded real projective variety. That is, we call a set in such a variety *extendably convex* if it is equal to the intersection of the variety with a convex body in the ambient projective space. We then present a technical lemma, related to the intersection of a polytope with a real embedded projective variety, which we use in the convexity proof in Section 3.5.

In Section 3.3 we define *exterior cyclic polytopes* and explore their combinatorial properties. In Section 3.4 we study a special class of their facets cut out by Schubert hyperplanes, which will be useful in Section 3.6. In Section 3.5 we connect exterior cyclic polytopes to amplituhedra. Concretely, we prove that the amplituhedron for $k = m = 2$ is equal to the intersection of the exterior cyclic polytope with the Grassmannian. Finally, in Section 3.6 we define a notion of duality for semialgebraic sets in the Grassmannian, and explore its concrete incarnation for the amplituhedron at $k = m = 2$.

## 3.1  Preliminaries on the twist map

From now on, we work over $\mathbb{R}$. Let $\mathbb{P}^N$ be the real projective space of dimension $N$ and let $\mathrm{Gr}(k, n)$ be the real Grassmannian of $k$-dimensional linear subspaces of $\mathbb{R}^n$. Let $Z$ be a real $(k + m) \times n$ matrix with positive maximal minors, where $n \geq k + m$. We identify the $i$-th column of $Z$ with a point in $\mathbb{P}^{k+m-1}$, which we denote by $Z_i$ or simply by the index $i$. We use $(i_1 \cdots i_r)$ to denote the $(r - 1)$-dimensional linear space in $\mathbb{P}^{k+m-1}$ spanned by the points $i_1, \ldots, i_r$ with $1 \leq r \leq k+m$. Then $(i_1 \cdots i_r)$ may be viewed as an element of $\mathrm{Gr}(r, k+m)$, corresponding to $Z_{i_1} \wedge \cdots \wedge Z_{i_r}$ via the Plücker embedding into $\mathbb{P}(\bigwedge^r \mathbb{R}^{k+m})$. Thus we can represent the (geometric) intersection of subspaces in $\mathbb{P}^{k+m-1}$ by an element in the exterior algebra $\mathbb{P}(\bigwedge \mathbb{R}^{k+m-1})$. That is, setting $t := N - r - s \geq 1$ we have that

$$
\begin{aligned}
(i_1 \cdots i_r) \cap (j_1 \cdots j_s) &= \sum_{I \in \binom{\{i_1, \ldots, i_r\}}{t}} \varepsilon(I, I^c) \langle I^c, j_1 \ldots j_s \rangle I \\
&= \sum_{J \in \binom{\{j_1, \ldots, j_s\}}{t}} \varepsilon(J, J^c) \langle i_1 \ldots i_r, J^c \rangle J.
\end{aligned}
\tag{3.3}
$$

Here $I^c := \{i_1, \ldots, i_r\} \setminus I$ and $\langle a_1 \ldots a_{k+m} \rangle$ denotes the corresponding maximal minor of $Z$. The symbol $\varepsilon(I, I^c) = \pm 1$ is equal to the sign of the permutation $(I, I^c)$ in one-line notation, where the elements of $I, I^c$ are listed in increasing order.

More generally, in the following we use angle brackets to denote the determinant of matrices obtained by stacking together the arguments in the brackets. We also used the shorthand notation $I = (i_{a_1} \cdots i_{a_t})$ if $I$ is equal to $\{a_1, \ldots, a_t\} \in \binom{\{i_1, \ldots, i_r\}}{t}$, and similarly for $J$. In the special case where $r + s = N$, we have that

$$
(i_1 \cdots i_r) \cap (j_1 \cdots j_s) := \langle i_1 \ldots i_r j_1 \ldots j_s \rangle.
\tag{3.4}
$$

This vanishes if and only if the subspaces $(i_1 \cdots i_r)$ and $(j_1 \cdots j_s)$ intersect non-trivially.

**Example 3.1.1.** Let $k + m = 4$. In $\mathbb{P}^3$, a plane $(i_1 i_2 i_3)$ intersects a line $(j_1 j_2)$ in a single point whenever $|\{i_1, i_2, i_3\} \cap \{j_1, j_2\}| \leq 1$, since $Z$ has nonzero maximal minors. This point is given by

$$
\begin{aligned}
(i_1 i_2 i_3) \cap (j_1 j_2) &= i_1 \langle i_2 i_3 j_1 j_2 \rangle - i_2 \langle i_1 i_3 j_1 j_2 \rangle + i_3 \langle i_1 i_2 j_1 j_2 \rangle \\
&= j_1 \langle i_1 i_2 i_3 j_2 \rangle - j_2 \langle i_1 i_2 i_3 j_1 \rangle .
\end{aligned}
\tag{3.5}
$$

Similarly, two distinct planes $(i_1 i_2 i_3)$ and $(j_1 j_2 j_3)$ intersect in the line

$$
\begin{aligned}
(i_1 i_2 i_3) \cap (j_1 j_2 j_3) &= (i_1 i_2) \langle i_3 j_1 j_2 j_3 \rangle - (i_1 i_3) \langle i_2 j_1 j_2 j_3 \rangle + (i_2 i_3) \langle i_1 j_1 j_2 j_3 \rangle \\
&= (j_1 j_2) \langle i_1 i_2 i_3 j_3 \rangle - (j_1 j_3) \langle i_1 i_2 i_3 j_2 \rangle + (j_2 j_3) \langle i_1 i_2 i_3 j_1 \rangle .
\end{aligned}
\tag{3.6}
$$

$\diamond$

We now move to the definition of the twist map on matrices. We denote by $\mathrm{Mat}_{>0}(k + m, n)$ the space of matrices with positive maximal minors, where we assume that $n \geq k + m$. We call an element in $\mathrm{Mat}_{>0}(k + m, n)$ a *positive matrix*. Let $Z$ be in $\mathrm{Mat}_{>0}(k + m, n)$ and denote by $Z_i \in \mathbb{R}^{k+m}$ the $i$-th column of $Z$. We use the *twisted cyclic* ordering on the index set $[n] := \{1, \ldots, n\}$, for which $Z_{n+i} = (-1)^{k-1} Z_i$. We identify $\bigwedge^{k+m-1} \mathbb{R}^{k+m}$ with $\mathbb{R}^{k+m}$ via the standard inner product. Consider then the vector in $\mathbb{R}^{k+m}$ defined by

$$
W_i := Z_{i-m+1} \wedge Z_{i-m+2} \wedge \cdots \wedge Z_i \wedge \cdots \wedge Z_{i+k-1} , \qquad i \in [n] .
\tag{3.7}
$$

Geometrically, we may think of $W_i$ as the normal to the hyperplane in $\mathbb{R}^{k+m}$ spanned by the vectors $Z_{i-m+1}, \ldots, Z_{i+k-1}$.

**Example 3.1.2.** Let $k = m = 2$. Then the vector $W_i \in \mathbb{P}^3_{\mathbb{R}}$ is normal to the plane $\bar{i} := (i-1\, i\, i+1)$, where the indices are taken modulo the twisted cyclic action described above. Moreover,

$$
\overline{ij} := \bar{i} \cap \bar{j} = W_i \wedge W_j \ \in \ \mathbb{P}(\overset{2}{\bigwedge} \mathbb{R}^4) .
\tag{3.8}
$$

According to the twistor notation, we identify $\overline{ij}$ with the line in $\mathbb{P}^3$ given by the intersection of the two planes $\bar{i}$ and $\bar{j}$. The lines (3.8) will be relevant when discussing the exterior cyclic polytope. $\diamond$

**Definition 3.1.3** (Twist map). We now define the *twist map* as

$$
\begin{aligned}
\tau \ : \ \mathrm{Mat}_{>0}(k + m, n) &\to \mathrm{Mat}(k + m, n) , \\
Z &\mapsto W ,
\end{aligned}
$$

where $W$ is the matrix with columns $W_1, \ldots, W_n$ as in (3.7).

The twist map for Grassmannians was first defined by Marsh and Scott [MS13]. An early incarnation appears in Berenstein, Fomin, and Zelevinsky [BFZ96], in the context of parameterizations of totally positive unipotent matrices. Up to sign changes and a cyclic

shift, the twist map in Definition 3.1.3 coincides with the *right twist* as defined by Muller and Speyer [MS17], but we will follow the convention of [GL20, Definition 4.1]. Muller and Speyer also define a left twist and show that the right and left twists are mutual inverse diffeomorphisms of $\mathrm{Mat}_{>0}(k+m, n)$; see [MS17, Theorem 6.7, Corollary 6.8]. In particular, $\tau$ maps a positive matrix to a positive matrix.

For $m = 4$ the twist map is related to the notion of *parity duality* in physics [ATT18], which has been generalized purely at the level of the amplituhedron for any $m$ [GL20].

## 3.2 Convexity and intersections with polytopes

The goal of this section is to define a notion of convexity for semialgebraic sets in real embedded projective varieties, and in particular for semialgebraic subsets of the Grassmannian. The following definition of semialgebraic set is used in [ABL17], where positive geometries were first defined. In Chapter 4 we will generalize the definition to semialgebraic sets in arbitrary real schemes, but for now we will not need this level of generality.

**Definition 3.2.1** (Semialgebraic set)**.** A *basic semialgebraic cone* $S$ in $\mathbb{R}^{N+1}$ is a subset defined by homogeneous equations and inequalities. A *semialgebraic cone* is a finite boolean combination of basic semialgebraic cones. A *semialgebraic set* in real projective space $\mathbb{P}^N$ is the image of a semialgebraic cone under $\pi : \mathbb{R}^{N+1} \setminus \{0\} \to \mathbb{P}^N$. Semialgebraic subsets of a projective variety $X$ are semialgebraic subsets of the ambient projective space lying in $X$.

We will study convexity in the projective setting. We call a subset $S \subset \mathbb{P}^N$ *convex* if it is of the form $\mathbb{P}(C)$ for some convex set $C \subset \mathbb{R}^{N+1}$, i.e. the image of $C$ under the projection map $\mathbb{R}^{N+1} \setminus \{0\} \to \mathbb{P}^N$, $x \mapsto [x]$. The notion of convex hull of a connected set is independent on the choice of affine chart on projective space, or equivalently on the choice of a hyperplane $H$ at infinity, provided that $H$ is disjoint from $S$, see [KS22, Lemma 2.2].

**Definition 3.2.2.** Let $S \subset \mathbb{P}^N$ be a semialgebraic set.

1. The set $S$ is *very compact* if there is a real hyperplane $H \subset \mathbb{P}^N$ such that $H \cap S = \emptyset$.

2. If $S$ is connected and very compact, then the *convex hull* of $S$ is

$$\mathrm{conv}(S) = \iota_H(\mathrm{conv}(\iota_H^{-1}(S)) \subset \mathbb{P}^N \,, \tag{3.9}$$

where $\iota_H : \mathbb{R}^N \cong \mathbb{P}^N \setminus H \hookrightarrow \mathbb{P}^N$ is the affine chart associated to $H$.

### 3.2.1 Extendable convexity

We now introduce a notion of convexity for semialgebraic sets in a real embedded projective variety $X \subset \mathbb{P}^N$, arising from convexity in the ambient projective space. The main case of interest to us is when $X = \mathrm{Gr}(k, n)$ in its Plücker embedding. This was first considered by Busemann in [Bus61].

**Definition 3.2.3.** Let $X \subset \mathbb{P}^N$ be a real variety and let $S \subset X$ be a connected, semialgebraic, and very compact set. We define the *convex hull of $S$ in $X$* to be

$$\text{conv}_X(S) := X \cap \text{conv}(S) . \tag{3.10}$$

We say that $S$ is *extendably convex* if $S = \text{conv}_X(S)$.

By the Tarski-Seidenberg Principle, taking convex hulls preserves semialgebraicity [BCR10, Section 1.4]. Therefore, $\text{conv}_X(S)$ is a semialgebraic set in $X$. Note that $S$ is extendably convex in $X$ if and only if it is equal to the intersection of $X$ with a convex set in $\mathbb{P}^N$.

Because this notion of convexity is inherited from that in projective space, we immediately obtain the following properties, which hold for the convex hull of cones in real vector spaces.

**Proposition 3.2.4.** *Let $X \subset \mathbb{P}^N$ be a real variety and let $S, T \subset X$ be connected, semialgebraic, and very compact sets. The following properties hold.*

1. *Monotonicity:* $S \subset T \implies \text{conv}_X(S) \subset \text{conv}_X(T)$.

2. *Idempotence:* $\text{conv}_X(\text{conv}_X(S)) = \text{conv}_X(S)$.

3. *Antiexchange: If $S$ is extendably convex, $x, y \notin S$, $x \neq y$ and $x \in \text{conv}_X(S \cup \{y\})$, then $y \notin \text{conv}_X(S \cup \{x\})$.*

If the variety $X$ is invariant under the action of a subgroup of $\text{PGL}(N+1)$, the automorphism group of $\mathbb{P}^N$, then we obtain an additional equivariance property for the convex hull in $X$. This is the case for the Grassmannian $\text{Gr}(k, n)$, which is a symmetric space given by a quotient of $\text{GL}(n)$. In this case, $\text{GL}(n)$ acts on $\mathbb{P}^N \cong \mathbb{P}(\bigwedge^k \mathbb{R}^n)$ with $N = \binom{n}{k} - 1$ via the exterior product of linear maps.

**Proposition 3.2.5.** *Let $X = \text{Gr}(k, n)$ and $S \subset X$ be a connected, semialgebraic, and very compact set. Then, the extendably convex hull is $\text{GL}(n)$-equivariant, namely*

$$\forall \, \sigma \in \text{GL}(n) \, : \, \text{conv}_X(\sigma(S)) = \sigma(\text{conv}_X(S)) .$$

*Proof.* We combine $\text{PGL}(N)$-equivariance of the convex hull in projective space, where $N = \binom{n}{k} - 1$, with the fact that $\text{Gr}(k, n)$ is stable under the action of $\text{GL}(n)$:

$$\sigma(\text{conv}_X(S)) = \sigma(\text{conv}(S) \cap \text{Gr}(k, n)) = \sigma(\text{conv}_X(S)) \cap \sigma(\text{Gr}(k, n))$$
$$= \text{conv}(\sigma(S)) \cap \text{Gr}(k, n) = \text{conv}_X(\sigma(S)) . \qquad \square$$

Given a subset $S \subset \mathbb{P}^N$, we denote the interior by $\text{int}(S)$, the closure by $\text{cl}(S)$ and the boundary of $S$ by $\partial S$, where the boundary is with respect to the Euclidean (quotient) topology on $\mathbb{P}^N$. Similarly, if $S \subset X$ for some subvariety $X \subset \mathbb{P}^N$, we denote the interior by $\text{int}_X(S)$, the closure by $\text{cl}_X(S)$ and the boundary by $\partial_X S$. All of this is relative to $X$, i.e. with respect to the subspace topology on $X$. It is a standard exercise in topology to show that $\text{cl}_X(S) = X \cap \text{cl}(S)$ and $\text{int}_X(S) = \text{int}((X \cap S) \cup X^c)$.

**Definition 3.2.6.** Let $S \subset X$ be a semialgebraic set in a real projective variety $X$. The *algebraic boundary* $\partial_a S$ of $S$ is the Zariski closure of $\partial_X S$ in $X$.

We first want to establish when the algebraic boundary of a semialgebraic set in a real projective variety is pure of codimension one. Throughout the following, let $X \subset \mathbb{P}^N$ be an irreducible real projective variety. We start from a purely topological definition.

**Definition 3.2.7.** A subset $S \subset X$ is called *regular in $X$* if it is contained in the closure in $X$ of its relative interior in $X$. In formulas,

$$S \subset \mathrm{cl}_X(\mathrm{int}_X(S)). \tag{3.11}$$

We define the *irregular points* of $S$ to be

$$\mathrm{cl}_X(\mathrm{int}_X(S)) \setminus S, \tag{3.12}$$

so that $S$ is regular in $X$ if and only if the set of its irregular points is empty.

For example, if $S \subset X$ is open in $X$, or if it is equal to the closure in $X$ of an open set in $X$, then $S$ is regular in $X$.

**Remark 3.2.8.** Suppose that $X \subset \mathbb{P}^N$ is non-degenerate, that is, that $X$ is not contained in any projective space of dimension smaller than $N$. Then, any subset $S \subset X$ with non-empty interior affinely spans $\mathbb{P}^N$, and in particular $\mathrm{conv}(S) \subset \mathbb{P}^N$ is full-dimensional.

**Example 3.2.9.** In projective space, each convex semialgebraic set $S$ with non-empty interior is regular, and so is its complement [Sin14, Remark 2.4]. This does not hold in general for extendably convex sets in real projective varieties. As an example, we work in an affine chart of $\mathbb{P}^3$ and take $X = \mathcal{V}(x^2 + y^2 + z^2 - 1) \subset \mathbb{R}^3$ to be the sphere of unit radius, and $S$ to be the convex hull in $\mathbb{R}^3$ of the union of the closed upper hemisphere of $X$ with a great arc segment in the lower hemisphere,

$$S = \mathrm{conv}(X \cap \{(x, y, z) : z \geq 0\} \cup \{X \cap \mathcal{V}(y, z)\}). \tag{3.13}$$

Then, $S$ is convex, and therefore $X \cap S$ is convex in $X$. However, $X \cap S$ is not regular in $X$, as the closure of the interior does not contain the arc segment. On the other hand, since $S$ is closed in $X$, the complement $X \setminus S$ is open and hence regular in $X$. ◇

The following is an extension of [RST24, Lemma 2.5]. The same proof can be applied to this setting since the argument is purely local.

**Lemma 3.2.10.** *Let $\emptyset \neq S \subset X$ be a semialgebraic set, regular in $X$, and suppose that $X \setminus S$ is non-empty and regular in $X$. Then each irreducible component of the algebraic boundary of $S$ has codimension one in $X$, i.e. $\partial_a S$ is a hypersurface in $X$.*

## 3.2.2 Intersections with convex polytopes

In this section we give a criterion for when a semialgebraic set $S$ in a real projective variety $X$ is equal to the intersection of $X$ with a convex set $P$ in the ambient projective space. Our main case of interest is when $X$ is the Grassmannian and $P$ is a projective polytope. The results presented here will be crucial in proving Theorem 3.5.7. We start with some elementary results from topology.

**Lemma 3.2.11.** *Let $X$ be a topological space and $U, V \subset X$ open subsets such that $U$ is connected, $\emptyset \neq V \subseteq U$ and $\partial V \subseteq \partial U$. Then $V = U$.*

*Proof.* Since $U$ is open, $U \cap \partial U = \emptyset$, and therefore by assumption also $U \cap \partial V = \emptyset$. Then

$$U = V \cup (U \setminus \overline{V}) \cup (U \cap \partial V) = V \cup (U \setminus \overline{V}). \tag{3.14}$$

Since $U$ is connected and $V$ is nonempty, it follows that $U \setminus \overline{V} = \emptyset$. Hence $U = V$. $\qquad\square$

Let us now refine this result for our setting of interest.

**Lemma 3.2.12** (Intersection with polytope). *Let $S$ be a regular closed semialgebraic set in a real irreducible smooth non-degenerate projective variety $X \subset \mathbb{P}^N$. Let $P$ be a full-dimensional projective convex polytope in $\mathbb{P}^N$. Assume that the following conditions hold, where all inclusions are in $\mathbb{P}^N$:*

1. *$X \cap P$ is regular in $X$,*

2. *$X \cap \operatorname{int}(P)$ is connected,*

3. *$\emptyset \neq \operatorname{int}_X(S) \subseteq \operatorname{int}(P)$,*

4. *$\partial_a S \subset \partial_a P$.*

*Then*

$$S = X \cap P, \quad \text{and} \quad \partial_X S = X \cap \partial P. \tag{3.15}$$

*In particular, $S$ is extendably convex.*

*Proof.* We first show that $\partial_X(X \cap P) = X \cap \partial P$. It is a simple exercise to show that $\partial_X(X \cap P) \subset X \cap \partial P$, so it suffices to show the reverse inclusion. We argue by contradiction. Let $p$ be a point in $X \cap \partial P$, and suppose that $p$ is not in $\partial_X(X \cap P)$. Then $p$ is in the interior of $X \cap P$, and $X \cap \partial P$ has the subspace topology in $X \cap P$, so there is an open neighborhood of $p$ in $X$ contained in $X \cap \partial P$. Since $X$ is irreducible, it is equal to the Zariski closure of any of its open subsets. Thus $X$ must be contained in $\partial_a P$. Since $P$ is a polytope, $\partial_a P$ is a union of hyperplanes in $\mathbb{P}^N$. But $X$ is non-degenerate, so it cannot be contained in a hyperplane.

We now prove the inclusion $\partial_X S \subset \partial P$. This follows from $\partial_a P \cap P = \partial P$, as $P$ is a projective convex polytope. In fact, by assumption 3 and the fact that $S$ is regular we

have that $\partial_X S = \partial_X \operatorname{int}_X(S) \subset P$. On the other hand, by assumption 4 we have that $\partial_X S \subset \partial_a S \subset \partial_a P$, and intersecting with $P$ we obtain that $\partial_X S \subset \partial_a P \cap P = \partial P$.

We now prove the result. Let $V = \operatorname{int}_X(S)$ and $U = \operatorname{int}(X \cap P)$, which are open subsets of $X$. Since $S$ is regular in $X$, $\partial_X S = \partial_X V$. Similarly, by assumption 1 it follows that $\partial_X(X \cap P) = \partial_X U$. Then, by assumption 3 we have that $\partial V \subset X \cap \partial P = \partial_X U \subset \partial U$. Therefore, we can apply Lemma 3.2.11 to deduce that $U = V$. In particular, $\partial U = \partial V$. Moreover, by the fact that $S$ is regular, together with assumption 1, we deduce that $S = \operatorname{cl}_X(V) = \operatorname{cl}_X(U) = X \cap P$. $\qquad\square$

We now characterize a subset of the potentially irregular points in the intersection of a projective variety with a polytope. This will be useful later when checking the assumptions of Lemma 3.2.12.

**Lemma 3.2.13.** *Let $X \subset \mathbb{P}^N$ be a real projective variety of codimension $r$ and let $P \subset \mathbb{P}^N$ be a full-dimensional projective convex polytope in $\mathbb{P}^N$ such that $X \cap \operatorname{int}(P) \neq \emptyset$. Let $F \subset \partial P$ be a face of $P$ of dimension greater or equal than $r$. Then, the irregular points of $X \cap P$ in the relative interior of $F$ in $X$ are contained in the singular locus of $X \cap F$, i.e.*

$$(\operatorname{cl}_X(\operatorname{int}_X(X \cap P)) \setminus (X \cap P)) \cap \operatorname{relint}_X(F) \subset \operatorname{Sing}(X \cap \overline{F}) \cap \operatorname{relint}_X(F) , \qquad (3.16)$$

*where $\overline{F}$ denotes the Zariski closure of $F$ in $\mathbb{P}^N$ and $\operatorname{relint}_X(F) := \operatorname{int}_{\overline{F}}(X \cap F)$.*

*Proof.* Let $p \in X \cap \operatorname{relint}_X(F)$. Note that $p$ is regular in $X \cap F$ if and only if $X$ and $F$ intersect transversally at $p$. By the assumption on dimensions, this is equivalent to $\dim(T_p F) + \dim(T_p X) = N$. In turn, since $p \in \operatorname{relint}_X(F)$ and by dimension reasons, this is equivalent to any small open ball in $X$ around $p$ intersecting $\operatorname{int}(P)$, i.e. to $p$ being regular. $\qquad\square$

This criterion is particularly useful if $X$ is a hypersurface, and if one understands the singular locus of the intersection of $X$ with each face of $P$. This is the case in Section 3.5.

## 3.3 Exterior cyclic polytopes

We introduce the exterior cyclic polytope associated to a totally positive matrix. These polytopes are of independent interest for combinatorics, beyond applications to amplituhedra. We study the facet structure, $f$-vector, and dependence on the input matrix.

**Definition 3.3.1.** Fix $Z \in \operatorname{Mat}_{>0}(k + m, n)$ with $n \geq k + m$. We define the *$k$-th exterior cyclic cone of type $(m, n)$* as

$$\widehat{C}_{k,m,n}(Z) := \operatorname{conv}\left(\{Z_{i_1} \wedge \cdots \wedge Z_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}\right) \subset \bigwedge^k \mathbb{R}^{k+m} . \qquad (3.17)$$

We define the *$k$-th exterior cyclic cone of type $(m, n)$* as $C_{k,m,n}(Z) := \mathbb{P}\,\widehat{C}_{k,m,n}(Z)$.

We use the indexing $k + m$ because the exterior polytope is empty for $m < 0$, and also to match the amplituhedron convention. We note the following.

**Lemma 3.3.2.** *The polytope $C_{k,m,n}(Z)$ is the image in $\mathbb{P}(\bigwedge^k \mathbb{R}^{k+m})$ of the standard simplex in $\mathbb{P}(\bigwedge^k \mathbb{R}^n)$ under the linear map $\bigwedge^k Z$.*

In particular, $C_{k,m,n}(Z)$ is a projective polytope. It is obtained by projectivizing the convex hull of the vectors

$$(i_1 \ldots i_k) := Z_{i_1} \wedge \cdots \wedge Z_{i_k}, \qquad 1 \leq i_1 < \cdots < i_k \leq n. \tag{3.18}$$

Each such vertex is the projection of $e_I := e_{i_1} \wedge \ldots \wedge e_{i_k}$. The facets of the exterior cyclic polytope are certain hyperplanes in $\bigwedge^k \mathbb{R}^{k+m}$ spanned by a subset of the vertices. They are certain flats of rank $\binom{k+m}{k} - 1$ in the linear matroid of the matrix $\wedge^k Z$. These sub-maximal flats are called *hyperplanes* of the matroid. It will often be convenient to study this matroid.

**Definition 3.3.3.** The *wedge power matroid* $W_{k,m,n}$ is the linear matroid of $\wedge^k Z$ for $Z$ a generic $(k + m) \times n$ matrix.

**Example 3.3.4.** The columns of $\wedge^k Z$ will have dependencies. For example, let $Z_1, \ldots, Z_6$ be the columns of a $4 \times 6$ matrix. Then we may write $Z_1 = aZ_2 + bZ_3 + cZ_4 + dZ_5$ for some scalars $a, b, c, d$. Taking wedge product with $Z_1$ on both sides produces a linear relation between the columns $Z_1 \wedge Z_2$, $Z_1 \wedge Z_3$, $Z_1 \wedge Z_4$, and $Z_1 \wedge Z_5$ of $\wedge^2 Z$. $\diamond$

We note that starting from $m = 2$, the linear matroid of $\wedge^k Z$ depends not only on the matroid of $Z$ but also on the matrix $Z$ itself, as we will see in Subsection 3.3.3. However, the matroid of $\wedge^k Z$ is still constant for a generic choice of $Z$; indeed, the vanishing of minors of $\wedge^k Z$ defines a Zariski closed subset of the set of $(k + m) \times n$ matrices. Thus Definition 3.3.3 makes sense.

We will see in Subsection 3.3.3 that the combinatorial type of $C_{k,m,n}(Z)$ varies as $Z$ varies over totally positive matrices. However, we may still prove some independence of $Z$.

**Lemma 3.3.5.** *The oriented matroid of $\wedge^k Z$, and thus the combinatorial type of the exterior cyclic polytope, depends only on the image of $Z$ in $\mathrm{Gr}_{>0}(k + m, n)$.*

*Proof.* The oriented matroid of $\wedge^k Z$ is given by the sign of the maximal minors of $\wedge^k Z$. Each such minor is a polynomial in the matrix entries of $Z$. We claim each such polynomial is invariant under the $\mathrm{SL}(k + m)$ action on $Z$. Indeed, any matrix $A \in \mathrm{SL}(k + m)$ induces an endomorphism $\wedge^k A$ of $\bigwedge^k \mathbb{R}^{k+m}$ by acting on each tensor factor. The determinant of $\wedge^k A$ is a power of $\det A$, so for any $\binom{k+m}{k} \times \binom{k+m}{k}$ submatrix $W$, we have $\det(\wedge^k A \cdot W) = \det(W)$. $\square$

The relationship between the oriented matroid of a set of points and the convex hull of said points is described in [Zie+99, Remark 3.7.3]. Oriented matroids have many cryptomorphic axiomatizations, including one in terms of *covectors* given in Section 2.4. As a reminder, the covectors of the oriented matroid coming from a point configuration are signed sets

$Y = (Y^+, Y^-)$ such that there exists a hyperplane $H$ with $Y^+ = Y \cap H^+$ and $Y^- = Y \cap H^-$. Thus a facet of the convex hull of these points is exactly the span of a set of points $Y$ such that the signed set $(Y, \varnothing)$ is a covector. This shows that the oriented matroid of $\wedge^k Z$ determines the combinatorial type of the exterior cyclic polytope $C_{k,m,n}(Z)$.

### 3.3.1 The case $m = 0$

Here $\wedge^k Z$ is a $1 \times \binom{n}{k}$ matrix whose entries are the maximal minors of $Z$. For a general matrix, these minors are nonzero; thus the matroid $W_{k,0,n}(Z)$ is equal to the uniform matroid $U(1, \binom{n}{k})$. The exterior cyclic polytope in this case is equal to the single point $\mathbb{P}^0$.

### 3.3.2 The case $m = 1$

The case $m = 1$ is already interesting. In this case, the exterior cyclic polytope lives in $\mathbb{P}(\bigwedge^k \mathbb{R}^{k+1}) = (\mathbb{P}^k)^\vee$. The columns of $Z$ yield a configuration of points in $\mathbb{R}^{k+1}$ or in $\mathbb{P}^k$.

**Definition 3.3.6.** Let $M$ be a point configuration in $\mathbb{R}^k$. The *discriminantal arrangement* of $M$ is the hyperplane arrangement consisting of all hyperplanes spanned by points in $M$.

**Proposition 3.3.7.** *The matroid $W_{k,1,n}$ is the linear matroid of the discriminantal arrangement of $n$ general points in $\mathbb{R}^{k+1}$.*

*Proof.* Each column $Z_I = Z_{i_1} \wedge \ldots \wedge Z_{i_k}$ in the $(k+1) \times n$ matrix $\wedge^k Z$ is the normal vector to a hyperplane in $\mathbb{R}^{k+1}$ passing through $k$ of the $n$ points. The matroid of a hyperplane arrangement is exactly defined as the linear matroid of the normal vectors. $\square$
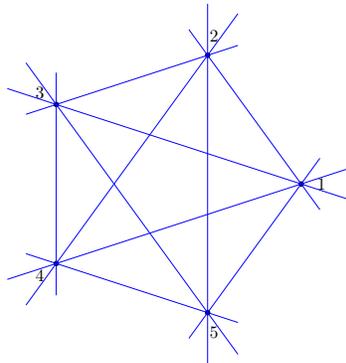


Figure 3.1: The discriminantal arrangement of five general points in $\mathbb{P}^2$

For example, the matroid $W_{k,1,k+2}$ is the graphic matroid of the complete graph $K_{k+2}$. Equivalently, it is the discriminantal arrangement of $k + 2$ general points in $\mathbb{R}^{k+1}$. This hyperplane arrangement is combinatorially equivalent to the braid arrangement.

For $m = 1$, the exterior cyclic polytope $C_{k,1,n} \subset (\mathbb{P}^k)^\vee$ is a $k$-simplex. The amplituhedron, as we have seen in the previous chapter, has the following concrete description: it is the union of bounded regions of the hyperplane arrangement whose normals are the columns of $\wedge^k Z$ [KW19]. For $k = 3$ and $n = 5$ the exterior cyclic polytope and the amplituhedron live in dual projective space $(\mathbb{P}^2)^\vee$ and are depicted in Figure 3.2. Notice that projective duality exchanges the five points in Figure 3.1 with the five lines in Figure 3.2.
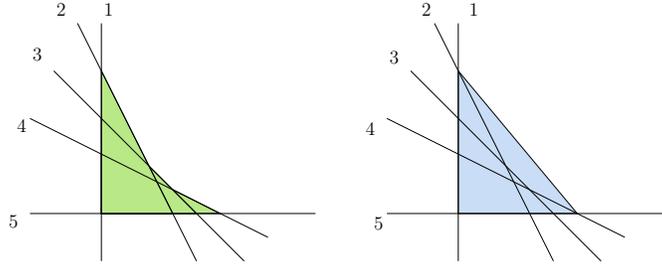


Figure 3.2: Amplituhedron (left) and exterior cyclic polytope (right) in $(\mathbb{P}^2)^\vee$ for $k = 3, m = 1, n = 5$.

### 3.3.3 The case $m = k = 2$ and $n = 6$

We present an extended example. Fix real numbers $a < b < c < d < e < f$. This gives a positive matrix

$$
Z \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e & f \\ a^2 & b^2 & c^2 & d^2 & e^2 & f^2 \\ a^3 & b^3 & c^3 & d^3 & e^3 & f^3 \end{pmatrix}. \tag{3.19}
$$

The convex hull of the columns of $Z$ is the cyclic polytope $C_{4,6}$, which is the convex hull of the union of two triangles meeting in a unique point in the interiors. Its $f$-vector is $(6, 15, 18, 9)$. Then $\wedge^2 Z$ is given by the $6 \times 15$ matrix whose entries are the $2 \times 2$ minors of $Z$:

$$
\begin{pmatrix}
a - b & a - c & a - d & a - e & \cdots & d - f & e - f \\
a^2 - b^2 & a^2 - c^2 & a^2 - d^2 & a^2 - e^2 & \cdots & d^2 - f^2 & e^2 - f^2 \\
a^3 - b^3 & a^3 - c^3 & a^3 - d^3 & a^3 - e^3 & \cdots & d^3 - f^3 & e^3 - f^3 \\
a^2 b - a b^2 & a^2 c - a c^2 & a^2 d - a d^2 & a^2 e - a e^2 & \cdots & d^2 f - d f^2 & e^2 f - e f^2 \\
a^3 b - a b^3 & a^3 c - a c^3 & a^3 d - a d^3 & a^3 e - a e^3 & \cdots & d^3 f - d f^3 & e^3 f - e f^3 \\
a^3 b^2 - a^2 b^3 & a^3 c^2 - a^2 c^3 & a^3 d^2 - a^2 d^3 & a^3 e^2 - a^2 e^3 & \cdots & d^3 f^2 - d^2 f^3 & e^3 f^2 - e^2 f^3
\end{pmatrix}.
$$

The (cone over the) exterior cyclic polytope is the convex hull of the columns of $\wedge^2 Z$ in $\bigwedge^2 \mathbb{R}^4$. We identify $\wedge^2 Z$ with the matrix that is obtained by dividing the columns by their common factors, $a - b, a - c, a - d, a - e, \ldots, d - f, e - f$. The resulting matrix has $\binom{16}{5} = 5005$

maximal minors, each of which is a homogeneous polynomial of degree 12. Of these minors, 1660 are zero and the other 3345 are nonzero. These bases come in 12 symmetry classes. We identify the ground set $\binom{[6]}{2}$ with edges of a complete graph on 6 vertices, and depict these bases in Figure 3.3. In all classes but one, the sign of the minor is fixed by the ordering of
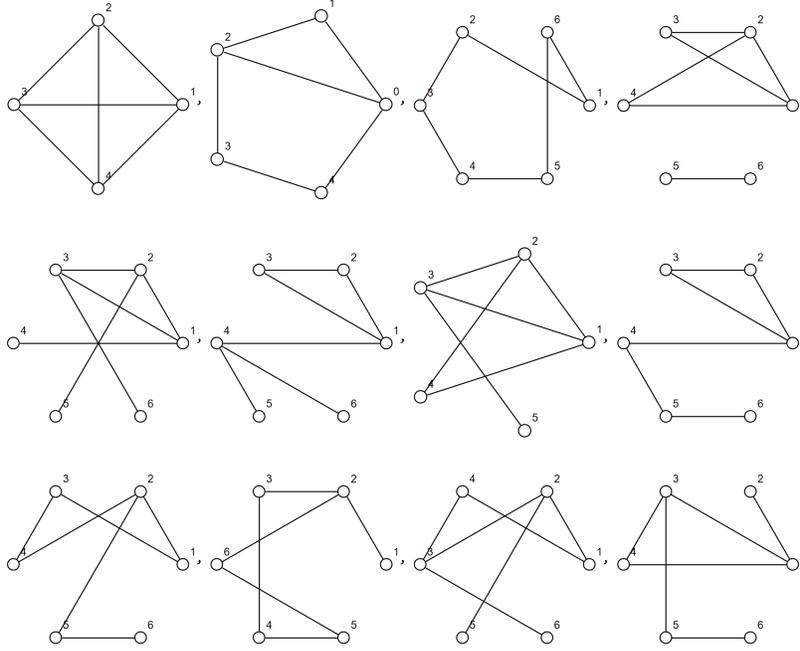


Figure 3.3: Bases of $W_{2,2,6}$

$a, b, c, d, e, f$. The only exception is the *cycle*:

$$[16, 12, 23, 34, 45, 56] = (a-c)(a-d)(a-e)(b-d)(b-e)(b-f)(d-f)(c-e)(c-f)$$
$$\cdot\, (abd - abe - acd + acf + ade - adf + bce - bcf - bde + bef + cdf - cef).$$

Here the minor has a cubic factor with 12 terms which can have any sign. For instance, fix $(a, b, c, d, e) = (1, 3, 4, 7, 8)$. Then the cubic seen above is positive for $f > 47/5$, zero for $f = 47/5$, and negative for $f < 47/5$. This shows that both the matroid and the oriented matroid of $\wedge^2 Z$ may change as $Z$ varies over positive matrices.

The combinatorics of $C_{k,m,n}(Z)$ also changes. For $f = 9$, our 5-dimensional polytope has $f$-vector $(15, 75, 143, 111, 30)$. Among the 30 facets, there are 18 simplices and six double pyramids over the pentagons, like $\{12, 13, 14, 15, 16, 23, 56\}$. The 18 simplex facets include

$$\{12, 23, 34, 45, 56\}, \ \{12, 23, 34, 56, 16\}, \ \{12, 16, 34, 45, 56\}.$$

For $f = 47/5$, these three simplices lie in a hyperplane, and they are replaced by one facet

$$\{12, 23, 34, 45, 56, 16\},$$

which is a cyclic polytope $C_{4,6}$. And, for $f > 47/5$, this is replaced by three other simplex facets:

$$\{12, 16, 23, 34, 45\}, \ \{12, 16, 23, 45, 56\} \ \{16, 23, 34, 45, 56\}.$$

The $f$-vector is the same for $f < 47/5$ and $f > 47/5$.

### 3.3.4   The case $m = k = 2$

When $k = 2$, the ground set of the matroid $W_{2,m,n}$ may be identified with the edges of a complete graph on $n$ labeled vertices. This matroid has been studied in the contexts of graph connectivity, graph rigidity, and algebraic statistics [Bra+24; CS23]. In particular, setting $d = n - m - 2$, the matroid of $\wedge^2 Z$ is equal to the dual of the *hyperconnectivity matroid* $\mathcal{H}_d(n)$ by [Bra+24, Theorem 1.1]. This matroid was introduced by Kalai to understand highly connected graphs [Kal85], and has rank $dn - \binom{d+1}{2}$ by Property 1 of [Kal85]. Thus the rank of its dual is indeed $\binom{n}{2} - dn + \binom{d+1}{2} = \binom{n-d}{2} = \binom{k+m}{2}$. The hyperconnectivity matroid $\mathcal{H}_d(n)$ is also equal to the algebraic matroid of $n \times n$ skew-symmetric matrices of rank at most $d$ [CS23, Proposition 3.1]. A characterization of the graphs which represent independent sets in $\mathcal{H}_d(n)$ is known only for $d = 2$ [Ber17].

To simplify combinatorial analyses in the hyperconnectivity matroid, we introduce the operations of cutting and gluing on graphs.

**Definition 3.3.8** (Cutting and gluing). Let $G$ be a simple undirected graph on $n$ vertices.

1. Let $e = uv$ be an edge in $G$. The $\text{Cut}(G, e, v)$ is the graph on $n + 1$ vertices obtained by introducing a new vertex $v'$ and replacing $uv$ with $uv'$.

2. Let $u$ and $v$ be vertices of $G$ with distance at least three. Then $\text{Glue}(G, u, v)$ is the graph on $n - 1$ vertices obtained by identifying $u$ and $v$ in the list of edges.

The distance condition for gluing guarantees that the new graph will be simple. Both operations preserve the number of edges. Furthermore, they are inverses: indeed, $\text{Glue}(\text{Cut}(G, e, v), v, v') = G$. We observe the following additional property.

**Lemma 3.3.9.** *Let $G$ be a simple undirected graph on $n$ vertices.*

1. *If $G$ is an independent set in $W_{2,m,n}$ then any cutting of $G$ is an independent set in $W_{2,m,n+1}$.*

2. *If $G$ is a dependent set in $W_{2,m,n}$ then any gluing of $G$ is a dependent set in $W_{2,m,n-1}$.*

For example, all circuits in $W_{2,2,n}$ may be obtained by gluings of the three circuits in Figure 3.4. Lemma 3.3.9 allows us to display the circuit data for all $n$ in a compact form.

The $f$-vector of the exterior cyclic polytope is given in Table 3.1 for $k = m = 2$ and $n$ small. We checked computationally with many positive $Z$ by taking a random Vandermonde matrix, computing its $LU$ decomposition, normalizing $U$, and setting $Z = U^T D L$ where $D$
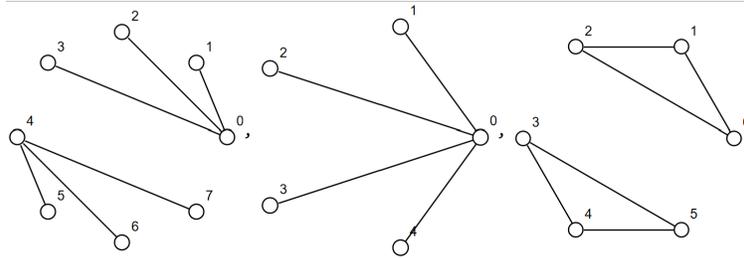
Figure 3.4: Circuits in $W_{2,2,n}$ up to gluing

is the diagonal of the first $k \times k$ minor of $U$. Beginning with $n = 7$, Vandermonde matrices themselves (before reversing the $LU$ decomposition) do not give a random enough sample; all exterior cyclic polytopes of Vandermonde matrices which we computed have 72 facets for $n = 7$, rather than the general facet count of 82. The data in Table 3.1 should be regarded as a conjecture that, though the combinatorial type changes as $Z$ varies, the $f$-vector is constant.

$$
\begin{array}{rlcccccc}
n = 5 & : & 10 & 35 & 55 & 40 & 12 & 1 \\
n = 6 & : & 15 & 75 & 143 & 111 & 30 & 1 \\
n = 7 & : & 21 & 147 & 328 & 282 & 82 & 1 \\
n = 8 & : & 28 & 266 & 664 & 616 & 192 & 1 \\
n = 9 & : & 36 & 450 & 1217 & 1191 & 390 & 1
\end{array}
$$

Table 3.1: The $f$-vectors for $n = 5$ through $n = 9$

The rest of this subsection will be dedicated to the proof of the following theorem. Let $S_n$ act on the matrix $Z$ by permuting the columns.

**Theorem 3.3.10.** *The matroid of $\wedge^2 Z$ for $Z \in \mathrm{Mat}_{>0}(4, n)$ is equal to the wedge power matroid $W_{2,2,n}$ outside of the closed locus where the polynomial $\det[Z_{12}\ Z_{23}\ Z_{34}\ Z_{45}\ Z_{56}\ Z_{16}]$ or one of its permutations is zero.*

In particular, this shows that the combinatorial type of $C_{2,2,n}(Z)$ is constant on each region in the complement of the hypersurfaces described in Theorem 3.3.10. The polynomial in the theorem may also be written in Plücker coordinates of $Z$ as

$$
p_{1234}p_{1356}p_{2456} - p_{1235}p_{1346}p_{2456} + p_{1235}p_{1246}p_{3456} \, . \tag{3.20}
$$

If one instead takes maximal minors $q_{ij}$ of any matrix whose rowspan is the kernel of $Z$ and applies Plücker relations, this polynomial obtains the simpler form $q_{12}q_{34}q_{56} - q_{23}q_{45}q_{16}$. This expression appears in physics in the algebraic prefactor of the six-dimensional scalar hexagon integral, see [DDH11, Section 2.5, Equation 42].

The proof of Theorem 3.3.10 is a computational check, which we present as follows. The question we must answer is: for which bases $\{i_1 j_1, \ldots, i_6 j_6\}$ in $W_{2,2,n}$ does there exist a positive matrix $Z$ with $Z_{i_1} \wedge Z_{j_1}, \ldots, Z_{i_6} \wedge Z_{j_6}$ dependent? We call a basis where such $Z$ exists *dynamic*, and a basis without this property *static*. For example, we saw in Subsection 3.3.3 that the basis $\{12, 23, 34, 45, 56, 16\}$ is dynamic.

**Lemma 3.3.11.** *Suppose that the basis represented by a graph $G$ is static. Then any basis obtained by cutting $G$ is static.*

*Proof.* We prove the contrapositive. Suppose there exists a positive matrix $Z$ for which the edges of $\mathrm{Cut}(G, e, v)$ represent a set of dependent columns in $\wedge^k Z$. This means we have some $Z_{iv}, Z_{jv'}$ among these dependent columns. If $v = v'$, then the columns remain dependent. $\square$

These operations give a poset structure to the 47 combinatorial types of bases of $W_{2,2,n}$, where $G > G'$ if $G'$ is obtained by cutting $G$. The poset and bases are pictured in Appendix A.

*Proof of Theorem 3.3.10.* In the poset of bases, the $K_4$ and "house" graph (next to the $K_4$ in Figure 3.3) dominate every basis except for the 6-cycle. By Lemma 3.3.11, if these two are static, then so is every basis other than the 6-cycle.

We note that the property of being static or dynamic is invariant under cyclic rotations of the vertex labels, but it is not invariant under the action of the full symmetric group. Thus we must show that for every permutation in $S_5$, the house graph is static. (The graph $K_4$ has the full symmetric group $S_4$ as its automorphism group, so it is enough to check once). By Lemma 3.3.5 the matroid of $\wedge^k Z$ only depends on the Plücker coordinates of $Z$. Thus it suffices to parameterize the positive Grassmannian $\mathrm{Gr}(4,5)$ by matrices

$$
Z = \begin{bmatrix} 1 & 0 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & 0 & x_2 \\ 0 & 0 & 1 & 0 & -x_3 \\ 0 & 0 & 0 & 1 & x_4 \end{bmatrix},
$$

with each $x_i$ positive. We then checked that, for each permutation $\sigma$ in $S_5$, the polynomial given by $\det[Z_{\sigma(12)}\ Z_{\sigma(13)}\ Z_{\sigma(23)}\ Z_{\sigma(34)}\ Z_{\sigma(45)}\ Z_{\sigma(15)}]$ is a positive combination of monomials in the $x_i$, and hence itself positive. $\square$

## 3.4 Schubert hyperplanes

### 3.4.1 A general characterization

Some of the facets of the exterior cyclic polytope are Schubert hyperplanes. In all known examples for $m = 1, 2$ and 4, only these Schubert hyperplanes contribute to the boundary of

the amplituhedron. With this motivation, in Subsection 3.4.1 we define a *Schubert hyperplane* of the wedge power matroid and give several equivalent characterizations in Lemma 3.4.2. We characterize all Schubert hyperplanes of $C_{2,2,n}(Z)$ in Theorem 3.4.5. In Section 3.4.2 we define a new polytope $\widetilde{C}_{2,2,n}(Z)$ by deleting the non-Schubert facets and relate its dual to the image of $Z$ under the twist map.

We will use $F$ to refer to a hyperplane in the matroid sense, and $H_F$ to refer to the corresponding projective linear space in $\mathbb{P}(\bigwedge^k \mathbb{R}^{k+m})$.

**Definition 3.4.1.** We call a hyperplane $F$ of $W_{k,m,n}$ a *Schubert hyperplane* if, for generic $Z$, the intersection $H_F(Z) \cap \mathrm{Gr}(k, k+m)$ is a Schubert divisor in $\mathrm{Gr}(k, k+m)$.

We check that the property of being Schubert is constant for an open set of matrices $Z$. Indeed, in the space $\mathbb{P}(\bigwedge^k \mathbb{R}^{k+m})^\vee$ of hyperplanes in $\mathbb{P}(\bigwedge^k \mathbb{R}^{k+m})$, a hyperplane is Schubert if and only if its coefficients satisfy the Plücker relations; see the proof of (1) to (2) in Lemma 3.4.2. Thus a non-Schubert hyperplane $H_F(Z)$ can only become Schubert for a Zariski closed set of matrices $Z$.

**Lemma 3.4.2** (Schubert hyperplanes). *Let $F = \{Q_1, \ldots, Q_l\}$ be a hyperplane in $W_{k,m,n}$. Let $r = k + m$. Then the following are equivalent.*

1. *$F$ is a Schubert hyperplane.*

2. *For $Z$ generic, the entries of the normal vector to $H_F(Z)$ satisfy the Plücker relations.*

3. *For $Z$ generic, there is a $(r - k - 1)$-space in $\mathbb{P}^{r-1}$ meeting all the $(k - 1)$-spaces $Z_{Q_1}, \ldots, Z_{Q_l}$.*

*Proof.* We first prove that (1) and (2) are equivalent. Suppose that $H$ is the Schubert divisor $\mathrm{Gr}(k, r)$ of all $(k - 1)$-spaces in $\mathbb{P}^{r-1}$ meeting a fixed $(r - k - 1)$-space $P$. We represent $P$ as the kernel of a $k \times r$ matrix, which we also denote $P$. We represent an arbitrary point $L$ of $\mathrm{Gr}(k, r)$ as the rowspan of a $k \times r$ matrix, which we also call $L$. Then $L$ lies on $H$ if and only if the determinant of the $k \times k$ matrix $P \cdot L^T$ vanishes. We use $p_I$ to denote maximal minors of $P$ and $q_I$ to denote maximal minors of $L$. The Cauchy-Binet formula gives us

$$\sum_{I \subset \binom{[r]}{k}} p_I q_I = 0.$$

This proves the equivalence of (1) and (2). To prove (3), choose $N = \binom{r}{k} - 1$ independent elements to span the hyperplane. By definition, they have a common transversal $(r - k - 1)$-space if and only if the hyperplane is Schubert. Since the other elements $Q_i$ lie in the same hyperplane, they must also meet this common transversal. $\square$

**Example 3.4.3.** There are four combinatorial types of hyperplanes in $W_{2,2,6}$, depicted in Figure 3.5. The number of facet-defining hyperplanes of $C_{2,2,6}$ of each type is $(6, 6, 3, 15)$,
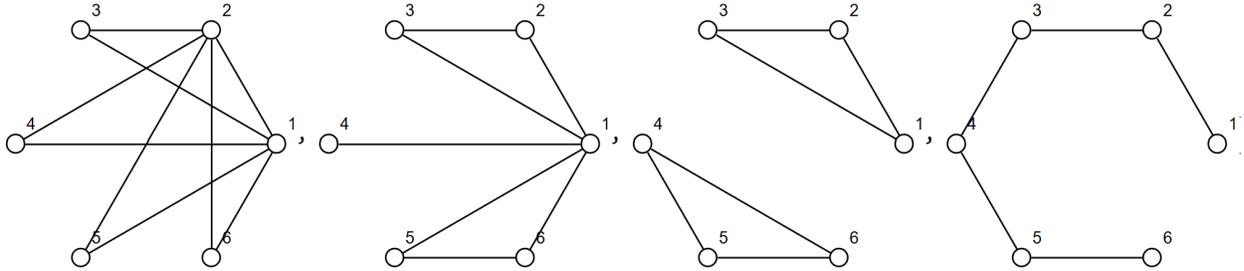
Figure 3.5: Hyperplanes of $W_{2,2,6}$

giving a total of thirty facets, as seen in Table 3.1. The first three types correspond to Schubert hyperplanes in $\mathrm{Gr}(2,4)$. That is, they are hyperplanes parameterizing lines in $\mathbb{P}^3$ meeting a fixed line. The common transversal line in each case is $12, (123) \cap (156)$ and $(123) \cap (456)$. The final graph in Figure 3.5 corresponds to a non-Schubert hyperplane. The corresponding facet of $C_{2,2,6}(Z)$ is a 5-simplex. ◇

One might conjecture from Example 3.4.3 that every hyperplane containing a circuit is a Schubert hyperplane. However, the following example shows that there may exist non-Schubert hyperplanes which are dependent sets in $W_{k,m,n}$.

**Example 3.4.4.** Let $k = 2$, $m = 3$ and $n = 15$. We label the columns of $Z$ by 1 through 9 and then $A$ through $F$. Then the set consisting of

$$\{12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D, 1E, 1F, 67, 89, AB, CD, EF\}$$

is a flat in $W_{2,3,15}$. It contains for example the circuit $\{12, 13, 14, 15, 16\}$. The span $H$ of the corresponding vectors in $\bigwedge^2 \mathbb{R}^5$ has dimension 9, so it is indeed a hyperplane. It contains all vertices of the form $Z_1 \wedge Z_*$. However, one may check computationally that the coefficients of $H$ do not satisfy the Plücker equations for $Z$ generic. The intuition is that given nine lines in $\mathbb{P}^4$, the condition that four of them meet at a point is not enough to guarantee a common transversal plane to all nine. ◇

**Theorem 3.4.5.** *The Schubert facets of the polytope $C_{2,2,n}(Z)$ are exactly the $\binom{n}{2}$ hyperplanes given by the vanishing of $\langle Y \overline{\overline{ij}} \rangle$ for $1 \leq i < j \leq n$, where $\overline{\overline{ij}} := (i - 1\,ii + 1) \cap (j - 1jj + 1)$. Furthermore, these Schubert facets intersect transversally in $\mathrm{Gr}(2,4)$ for every $Z \in \mathrm{Mat}_{>0}(4, n)$.*

*Proof.* Any Schubert hyperplane will contain one of the three circuits in Figure 3.4. Thus we only need to consider the following two cases:

1. three of the five lines lie in a common plane $P$, or

2. three of the five lines meet at a point $q$.

To complete the proof, it suffices to show that $F$ is a facet if and only if the transversal line $L$ furnished by Proposition 3.4.2 is of the form $\overline{ij}$ in each case. We show this explicitly by using the fact that if $\langle Y L\rangle$ is a facet, then $\langle Y L\rangle \geq 0$ for any vertex $Y$ of $C_{2,2,n}(Z)$. We spell out the details for case 1, and case 2 follows by a similar analysis.

Call the remaining two lines $D$ and $E$. Let us consider case 1 above, where $L = (P \cap D) \wedge (P \cap E)$. Here $P \cap D$ and $P \cap E$ are the intersection points of the lines $D$ and $E$ with $P$. Thus their span is indeed another line. Write $P = (ijk)$, with $i < j < k$. In the following we take all indices to be ordered modulo $n$. We first show that $P$ must involve consecutive indices. For that, we use eq. (3.5) and write

$$\langle Y L\rangle = \langle Y\,ij\rangle\langle ij\,(kD) \cap (kE)\rangle + \langle Y\,jk\rangle\langle jk\,(iD) \cap (iE)\rangle \\ + \langle Y\,ik\rangle\langle ik\,(jD) \cap (jE)\rangle. \tag{3.21}$$

Assume that $i + 1 < j < k - 1$. By plugging $Y = (i+1*)$ and $Y = (j+1*)$ in (3.21) for $* = i, j, k$ and requiring that (3.21) is nonnegative, we obtain that exactly two out of three coefficients of $\langle Y-\rangle$ in (3.21) must vanish. Since $i, j, k$ are distinct, and so are $D$ and $E$, this is a contradiction. Therefore, either $k = j + 1$, or $j = i + 1$. However, since $n \geq 4$, one can repeat the same argument to show that in fact there can be only one space between $i, j, k$, so that $P = (j - 1, j, j + 1) = \bar{j}$. Write now $D = (ab)$ and $E = (cd)$. Using again eq. (3.5) and applying the same reasoning as above, one finds that without loss of generality $c = b = a + 1$ and $d = a + 2$. Therefore, $L = \overline{j, a+1}$ has the desired form.

To prove that the Schubert hyperplanes intersect transversally, we use that the twist map preserves positivity of matrices. If $i, j, k, l$ are distinct, then $\langle \overline{ij}\overline{kl}\rangle$ is a maximal minor of $\tau(Z)$. If $Z$ is positive, then this minor is nonzero. Thus the lines $\overline{ij}$ and $\overline{kl}$ cannot intersect. $\qquad\square$

We now describe a subset of Schubert facets of $C_{k,m,n}(Z)$ believed to form part of the algebraic boundary of the amplituhedron. In [ATT18] the authors conjecture that the amplituhedron $\mathcal{A}_{k,m,n}(Z) \subset \mathrm{Gr}(k, k+m)$ has the following semialgebraic description:

$$(-1)^k\langle Y\,1\,(i_1 i_1 + 1)\ldots(i_{\frac{m-1}{2}} i_{\frac{m-1}{2}} + 1)\rangle,\ \langle Y\,(i_1 i_1 + 1)\ldots(i_{\frac{m-1}{2}} i_{\frac{m-1}{2}} + 1)n\rangle > 0 \quad m \text{ odd}, \tag{3.22}$$

$$\langle Y\,(i_1 i_1 + 1)\ldots(i_{\frac{m}{2}} i_{\frac{m}{2}} + 1)\rangle > 0 \quad m \text{ even}, \tag{3.23}$$

$$(\langle Y\,12\ldots m - 1m\rangle),\ \ldots,\ \langle Y\,12\ldots n - 1n\rangle)) \text{ has } k \text{ sign flips}. \tag{3.24}$$

To our knowledge, this semialgebraic description of the amplituhedron has been proven only for $m = 2$ in [PSW23, Theorem 5.1]. For this case, the algebraic boundary is known to consist of the $n$ Schubert divisors $\langle Y\,ii + 1\rangle = 0$. This has been proven for $k = 2$ in [RST24, Proposition 3.1], but a similar proof works for higher $k$. Furthermore, in [ELT25, Corollary 8.8] the authors showed that the algebraic boundary of the $m = 4$ amplituhedron is given by the Schubert divisors $\langle Y\,ii + 1jj + 1\rangle = 0$.

**Proposition 3.4.6.** *Equations* (3.22) *and* (3.23)*, for $m$ odd or even respectively, define facets of $C_{k,m,n}(Z)$.*

*Proof.* By positivity of $Z$, all vertices of $C_{k,m,n}$ lie on the same side of each hyperplane in (3.22) and (3.23). It remains to show that each hyperplane contains $\binom{k+m}{k} - 1$ vertices of $C_{k,m,n}$ in general position. For general $n$, the number of vertices on one such hyperplane is

$$\binom{n}{k} - \binom{n-m}{k}. \tag{3.25}$$

For $n = k + m$ this exactly equals $\binom{k+m}{k} - 1$. Since $n$ is at least $k + m$, it suffices to consider vertices labeled by subsets of $[k+m]$. These are linearly independent since $Z$ is positive. $\quad\square$

### 3.4.2 A duality for $m = k = 2$

We define a new polytope which only involves the Schubert hyperplanes. We are giving up some combinatorial control of the vertices for a better understanding of the facets.

Polyhedral cones have multiple presentations; they can appear as the convex hull of finitely many vectors, or as a subset of $\mathbb{R}^{N+1}$ defined by finitely many linear inequalities. We now work with the latter presentation. Generalizing our previous definition, we call a hyperplane in $\bigwedge^k \mathbb{R}^n$ a *Schubert hyperplane* if its projectivization is a Schubert hyperplane in $\mathbb{P}(\bigwedge^k \mathbb{R}^n)$.

**Definition 3.4.7.** The *Schubert exterior cyclic cone* is obtained from the exterior cyclic cone $\widehat{C}_{k,m,n}(Z)$ by deleting all facet inequalities corresponding to non-Schubert facets. In the case when the resulting cone is proper, we call its projectivation $\widetilde{C}_{k,m,n}(Z)$ in $\mathbb{P}(\bigwedge^k \mathbb{R}^n)$ the *kth Schubert exterior cyclic polytope of type* $(m,n)$.

**Example 3.4.8.** When $m = 1$ every hyperplane is Schubert, and hence $C_{k,1,n}(Z) = \widetilde{C}_{k,1,n}(Z)$. The same is true for $k = 1$. $\hfill\diamond$

In principle, when deleting facet inequalities one could end up with a non-proper cone, in which case its projectivization is not very compact. In our case, Proposition 3.4.6 tells us that $C_{k,m,n}(Z)$ has at least $\binom{n}{m/2}$ Schubert facets. When this number is at least $\binom{k+m}{k}$, the Schubert exterior cyclic cone in $\bigwedge^k \mathbb{R}^{k+m}$ corresponding to $Z$ is pointed. Indeed, the normal vectors span a space of dimension $\binom{n}{m/2}$. This implies that $\widetilde{C}_{k,m,n}(Z)$ is a projective polytope. In the case $k = m = 2$, Lemma 3.4.5 produces $\binom{n}{2}$ hyperplanes and $\widetilde{C}_{2,2,n}(Z)$ lives in $\mathbb{P}^5$, so we indeed get a polytope.

We study the Schubert exterior cyclic polytope $\widetilde{C}_{2,2,n}(Z)$ and its relation to $C_{2,2,n}(Z)$. For that, we find it useful to present a result appearing in [ATT18, Section 14], whose proof can be found in [Eve+24, Corollary 4.10]. In that reference it is shown that eq. (3.26) is a cluster variable for $\mathrm{Gr}_{>0}(4,n)$.

**Proposition 3.4.9** (Positive determinants for $k = m = 2$). *Let $Z \in \mathrm{Mat}_{>0}(4,n)$. Let $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$ be elements in $[n]$ such that $i_r \leq j_r$ for every $r = 1, 2, 3$ and*

*equality does not hold for all $r$. Let $a < b$ in $[n] \setminus ([i_1 + 1, \ldots, i_3 - 1] \cup [j_1 + 1, \ldots, j_3 - 1])$. Then,*

$$\det \begin{pmatrix} \langle a i_1 i_2 i_3 \rangle & \langle a j_1 j_2 j_3 \rangle \\ \langle b i_1 i_2 i_3 \rangle & \langle b j_1 j_2 j_3 \rangle \end{pmatrix} = \langle ab(i_1 i_2 i_3) \cap (j_1 j_2 j_3) \rangle \geq 0 \ . \tag{3.26}$$

The polytope $\widetilde{C}_{2,2,n}(Z)$ contains $C_{2,2,n}(Z)$ by definition. Moreover, by Theorem 3.4.5 and Proposition 3.4.9 we have that

$$\widetilde{C}_{2,2,n}(Z) = \{Y \in \mathbb{P}^5 \ : \ \langle Y \, \overline{ij} \rangle \geq 0 \, , \quad \forall \, 1 \leq i < j \leq n\} \, . \tag{3.27}$$

The polytope $\widetilde{C}_{2,2,n}$ is related to the dual polytope $C^*_{2,2,n}$ of $C_{2,2,n}$ as follows.

**Proposition 3.4.10.** *We have that*

$$\widetilde{C}_{2,2,n}(Z) = C_{2,2,n}(W)^* \, , \tag{3.28}$$

*where $W = \tau(Z)$ is the twist of $Z$, according to Definition 3.1.3.*

*Proof.* By definition of the exterior cyclic polytope, we have that $C_{2,2,n}(W) \subset \mathbb{P}^5$ is equal to the convex hull of the $\binom{n}{2}$ points $W_i \wedge W_j = \overline{ij}$. Note that $\overline{ii+1} = (ii+1)$, and hence by Proposition 3.4.9 we have that $\overline{ij}$ is an inward-pointing normal vector of $C_{2,2,n}(W)^* \subset \mathbb{P}^5$. Thus $C_{2,2,n}(W)^*$ is given by the right side of eq. (3.27), and hence it is equal to $\widetilde{C}_{2,2,n}(Z)$. □

By eq. (3.28) it follows that the vertices of $\widetilde{C}_{2,2,n}(Z)$ lying on $\mathrm{Gr}(2,4)$ correspond to Schubert facets of $C_{2,2,n}(W)$, which by Lemma 3.4.5 are projectively dual to $(ij)$ for $1 \leq i < j \leq n$. Note that here we also used that for $W = \tau(Z) \in \mathrm{Mat}_{>0}(4, n)$ we have that $\tau(W)_i \wedge \tau(W)_j = (ij)$ as elements in $\mathbb{P}(\bigwedge^2 \mathbb{R}^4)$, which can be easily verified. We have therefore shown that

$$\mathrm{Gr}(2,4) \cap \mathrm{vert}(\widetilde{C}_{2,2,n}) = \mathrm{vert}(C_{2,2,n}) \, , \tag{3.29}$$

where $\mathrm{vert}(P)$ denotes the set of vertices of the polytope $P$.

## 3.5 Convexity of amplituhedra

In the following, we fix a matrix $Z \in \mathrm{Mat}_{>0}(k+m, n)$ for $n \geq k+m$, and omit the dependence on $Z$ in the notation. Our first theorem concerns the convex hull of the amplituhedron in Plücker space. Note that the amplituhedron is connected, so by [KS22, Lemma 2.2] the convex hull is well-defined.

**Proposition 3.5.1.** *The convex hull of the amplituhedron $\mathcal{A}_{k,m,n}$ in $\mathbb{P}^{\binom{k+m}{k}-1}$ is equal to the exterior cyclic polytope $C_{k,m,n}$.*

*Proof.* Because the nonnegative Grassmannian is contained in the standard projective simplex, we immediately get the inclusion

$$\mathcal{A}_{k,m,n} \subset \mathrm{Gr}(k, k+m) \cap C_{k,m,n}. \tag{3.30}$$

Monotonicity of the convex hull gives $\mathrm{conv}(\mathcal{A}_{k,m,n}) \subset C_{k,m,n}$. For the other direction we work in an affine chart containing $C_{k,m,n}$, so that $C_{k,m,n}$ equals the (affine) convex hull of its vertices. Since the vertices of $C_{k,m,n}$ are contained in $\mathcal{A}_{k,m,n}$, we infer that $C_{k,m,n} \subset \mathrm{conv}(\mathcal{A}_{k,m,n})$ by monotonicity of the convex hull. $\square$

In the following, we want to determine which amplituhedra are extendably convex. In light of Proposition 3.5.1, this amounts to asking for which parameters the inclusion in eq. (3.30) is an equality.

**Question 3.5.2.** For which values of $k, m, n$ is $\mathcal{A}_{k,m,n}$ extendably convex?

Note that in phrasing this question we dropped the dependence on $Z \in \mathrm{Mat}_{>0}(k+m, n)$, but it is not clear if this is a reasonable assumption. We expect the answer to Question 3.5.2 to depend on $m$. For $m = 0$ the amplituhedron and the exterior cyclic polytope are both equal to $\mathbb{P}^0$, and therefore coincide. However, for the case of $m = 1$ it follows from the discussion in Section 3.3.2 that the amplituhedron is not extendably convex. In general, we expect that the amplituhedron is not extendably convex for odd $m$.

If the amplituhedron $\mathcal{A}_{k,m,n}$ is extendably convex, its boundary is equal to $\mathrm{Gr}(k, k+m) \cap \partial C_{k,m,n}$. In particular, the algebraic boundary would consist of only linear sections in the Grassmannian. It has been proven that this is the case for $m = 1, 2$ and $4$. However, for $m = 6$, the boundary of the amplituhedron contains higher degree components [Eve+25].

In this section we prove that the amplituhedron for $k = m = 2$ is extendably convex. For that, we need a sequence of lemmas to verify the assumptions in Lemma 3.2.12. We start from the regularity assumption.

**Lemma 3.5.3.** *The vertices* $\mathrm{vert}(C_{k,m,n})$ *are all regular points in* $\mathrm{Gr}(k, k+m) \cap C_{k,m,n}$.

*Proof.* This follows from the fact that $\mathrm{vert}(C_{k,m,n})$ is contained in $\mathcal{A}_{k,m,n}$ and the relative interior of $\mathcal{A}_{k,m,n}$ in $\mathrm{Gr}(k, k+m)$ is contained in $\mathrm{Gr}(k, k+m) \cap \mathrm{int}(C_{k,m,n})$. $\square$

**Lemma 3.5.4.** *The semialgebraic set* $\mathrm{Gr}(2, 4) \cap C_{2,2,n}$ *is regular in* $\mathrm{Gr}(2, 4)$. *The same is true for* $\mathrm{Gr}(2, 4) \cap \widetilde{C}_{2,2,n}$.

*Proof.* Since the projective dual variety of $\mathrm{Gr}_{\mathbb{C}}(2, 4)$ is equal to $\mathrm{Gr}_{\mathbb{C}}(2, 4)$, any hyperplane section of $\mathrm{Gr}(2, 4)$ is singular if and only if the hyperplane is Schubert. In Lemma 3.4.5 we characterized all Schubert hyperplanes of $C_{2,2,n}$. Each of these is projectively dual to a point $\overline{ij} \in \mathrm{Gr}(2, 4)$ for some $1 \leq i < j \leq n$. Each such Schubert divisor is singular only at one point, given by $\overline{ij}$. Therefore the singular points of Schubert hyperplanes of $C_{2,2,n}$ intersected with $\mathrm{Gr}(2, 4)$ are all $\overline{ij}$ for $1 \leq i < j \leq n$. We show that $\overline{ij}$ is in $\partial C_{2,2,n}$ if and

only if $j = i + 1$, modulo $n$, in which case $\overline{ij} = (ii + 1)$. In fact, if $|j - i| > 1$, then using (3.6) we compute

$$\langle \overline{ij}, \overline{i-1i+1} \rangle = \langle i - 1, i + 1, i, i + 2 \rangle \langle i, j - 1, j, j + 1 \rangle \langle i - 2, i - 1, i, i + 1 \rangle < 0. \quad (3.31)$$

By eq. (3.27) it follows that $\overline{ij} \notin C_{2,2,n}$ if $|j - i| > 1$. On the other hand, $(ii+1) \in \partial C_{2,2,n}$ and it is in fact a vertex of $C_{2,2,n}$, and hence a regular point of $\mathrm{Gr}(2,4) \cap C_{2,2,n}$ by Lemma 3.5.3.

Note that since $\mathrm{Gr}(2,4)$ has codimension one in $\mathbb{P}^5$, Lemma 3.2.13 applies to all faces of $C_{2,2,n}$ of dimension greater equal or than one. We also use that for every face $F$ of $C_{2,2,n}$ we have that

$$\mathrm{Sing}(X \cap \overline{F}) \subset \bigcup_i \mathrm{Sing}(X \cap H_i) = \{(ii + 1) \,:\, i = 1, \ldots, n\}, \quad (3.32)$$

Here $(ii + 1)$ denotes the point given by the vector $Z_i \wedge Z_{i+1}$, $\overline{F}$ denotes the Zariski closure of $F$ and the union ranges over all facet-defining hyperplanes $H_i$ of $C_{2,2,n}$. The last equality follows from the previous analysis. The proof is therefore concluded by Lemma 3.5.3.

The exact same argument works also for $\mathrm{Gr}(2,4) \cap \widetilde{C}_{2,2,n}$. The fact that the vertices are regular also in this case follows from the containment of $C_{2,2,n}$ in $\widetilde{C}_{2,2,n}$. $\qquad \square$

We now check the connectedness assumption of Lemma 3.2.12 in our setting. We first need the following result. Recall that $\overline{ij}$ denote the intersection of the two planes $(i-1ii+1)$ and $(j-1jj+1)$ in $\mathbb{P}^3$.

**Lemma 3.5.5.** *We have that*

$$\{Y \,:\, \langle Y\,ij \rangle > 0, \ \forall 1 \le i < j \le n\} \cap \{Y \,:\, \langle Y\,\overline{ij} \rangle > 0, \ \forall 1 \le i < j \le n\} = \emptyset, . \quad (3.33)$$

*where the two sets lie in* $\mathbb{P}^5$.

*Proof.* Consider the map $T$, which depends on $Z$, given by

$$T \,:\, \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n}{2}},$$
$$\sum_{1 \le a < b \le n} \langle Y\,ab \rangle\, e_{ab} \mapsto \sum_{1 \le a < b \le n} \langle Y\,\overline{ab} \rangle\, e_{ab}. \quad (3.34)$$

In coordinates, $T$ is given by eq. (3.6), namely

$$e_{ab} \mapsto e_{a-1,a}\langle a+1, b-1, b, b+1 \rangle + e_{a,a+1}\langle a-1, b-1, b, b+1 \rangle - e_{a-1,a+1}\langle a, b-1, b, b+1 \rangle. \quad (3.35)$$

Here $e_{ab} \in \mathbb{R}^{\binom{n}{2}} \cong \bigwedge^2 \mathbb{R}^n$ is identified with the wedge product of the standard basis vectors $e_a, e_b \in \mathbb{R}^n$. The statement follows if $T(\mathbb{R}^{\binom{n}{2}}_{>0})$ does not intersect the positive orthant $\mathbb{R}^{\binom{n}{2}}_{>0}$. For that, we show that there exists a hyperplane separating the two cones. We choose $v = \sum_{1 \le a < b \le n} v_{ab}\, e_{ab}$ such that $v_{ab} > 0$ if $|b - a| = 2$ and equal to zero otherwise. Then, by eq. (3.35) and the fact that $Z$ is positive, $T_{ab} \cdot v_{ab} \le 0$ for every $1 \le a < b \le n$. This means that the hyperplane normal to $v$ indeed separates $T(\mathbb{R}^{\binom{n}{2}}_{>0})$ and $\mathbb{R}^{\binom{n}{2}}_{>0}$. $\qquad \square$

**Lemma 3.5.6.** *The sets* $\mathrm{Gr}(2,4) \cap \mathrm{int}(C_{2,2,n})$ *and* $\mathrm{Gr}(2,4) \cap \mathrm{int}(\widetilde{C}_{2,2,n})$ *are both connected.*

*Proof.* Let us denote by $S$ one of the two sets in the statement, since the same argument works in both cases. Let $Y$ be a point in $S$. Recall that by eq. (3.27) we have that $\langle Y \overline{ij} \rangle > 0$ for every $1 \leq i < j \leq n$. Then, by Lemma 3.5.5 there exist $1 \leq i < j \leq n$ such that $\langle Y ij \rangle < 0$. We can assume without loss of generality that $i$ and $j$ are different than 1. We show that there exists a path in $S \cup \{(12)\}$ from $Y$ to $(12) \in \mathrm{Gr}(2,4)$. Consider the two-dimensional plane $P$ parameterized by $\alpha, \beta \in \mathbb{R}$ as $P(\alpha,\beta) := \alpha\,(12) + (ij) + \beta\,Y \in \mathbb{P}(\bigwedge^2 \mathbb{R}^4)$. We intersect $P$ with $\mathrm{Gr}(2,4)$. Since points in $\mathrm{Gr}(2,4)$ correspond to decomposable vectors in $\mathbb{P}(\bigwedge^2 \mathbb{R}^4)$, we obtain the equation

$$P(\alpha,\beta) \wedge P(\alpha,\beta) = \alpha \langle 12ij \rangle + \alpha\beta \langle Y12 \rangle + \beta \langle Yij \rangle = 0. \tag{3.36}$$

This is the defining equation of a curve in the plane $P$. We solve

$$\alpha(\beta) = \frac{-\beta \langle Yij \rangle}{\langle 12ij \rangle + \beta \langle Y12 \rangle}, \tag{3.37}$$

and define $\gamma(\beta) := P(\alpha(\beta), \beta)$. Then $\gamma$ is a parameterization of the curve $P \cap \mathrm{Gr}(2,4)$. Note that $\langle Y12 \rangle > 0$ and $\langle 12ij \rangle > 0$. Also, $\alpha, \beta > 0$ parameterize the (relative interior of the) convex hull of $(12)$, $(ij)$ and $Y$, which lies in $S$. Eq. (3.37) is strictly positive for every $\beta > 0$, which means that $\gamma(\beta)$ is in $S$ for $\beta > 0$. Then $\gamma(\beta)$ traces a continuous path in $S$ as $\beta$ varies in $(0, \infty)$. The endpoints are $\gamma(0) = (12)$ and $\lim_{\beta \to \infty} \gamma(\beta) = \lim_{\beta \to \infty} \frac{1}{\beta} P(\alpha(\beta), \beta) = Y$.

Above we showed that the set $S \cup \{(12)\}$ is connected. We now show that $S$ is also connected. Let $P$ be either $C_{2,2,n}$ or $\widetilde{C}_{2,2,n}$. Since $\mathrm{Gr}(2,4)$ is a manifold, we may take a local chart around $(12)$. We may choose the chart small enough such that the image of $S$ in the chart is a contractible set. Since $(12)$ lies on its boundary, $S$ is connected. $\qquad\square$

We now summarize some known topological properties of the amplituhedron. The amplituhedron $\mathcal{A}_{k,m,n}$ is closed in $\mathrm{Gr}(k, k+m)$ by definition. It is regular since it is the continuous image of $\mathrm{Gr}_{\geq 0}(k,n)$, which is regular because it is homeomorphic to a $k(m-k)$-dimensional closed ball [GKL22]. By the same argument, the interior of $\mathcal{A}_{k,m,n}$ is connected. We are now fully equipped to prove the main result of this section.

**Theorem 3.5.7** (Intersection result for $k = m = 2$)**.** *We have that*

$$\mathcal{A}_{2,2,n}(Z) = \mathrm{Gr}(2,4) \cap C_{2,2,n}(Z) = \mathrm{Gr}(2,4) \cap \widetilde{C}_{2,2,n}(Z), \qquad \forall\, Z \in \mathrm{Mat}_{>0}(4,n). \tag{3.38}$$

*Proof.* This follows from Lemma 3.2.12 by taking $X = \mathrm{Gr}(2,4)$, $\mathcal{A} = \mathcal{A}_{2,2,n}(Z)$ and $P = C_{2,2,n}(Z)$ or $P = \widetilde{C}_{2,2,n}(Z)$, respectively. As discussed above, the amplituhedron is a regular semialgebraic set and its relative interior is connected. Assumption 1 of Lemma 3.2.12 follows from Lemma 3.5.4, assumption 2 from Lemma 3.5.6, assumption 3 is immediate and assumption 4 from [RST24, Proposition 3.1] and Proposition 3.4.6. $\qquad\square$

**Corollary 3.5.8.** *The amplituhedron $\mathcal{A}_{2,2,n}(Z)$ is extendably convex for all $Z \in \mathrm{Mat}_{>0}(4,n)$.*

## 3.6 Extendable dual amplituhedra

In this section we define a notion of convex duality for semialgebraic sets in a real projective variety and later apply it to define a notion of dual amplituhedron.

### 3.6.1 Extendable duality

Through the following section, let $X \subset \mathbb{P}^N$ be a real projective variety.

**Definition 3.6.1.** Let $S \subset X$ be a connected, semialgebraic, and very compact set. We define the *extendable convex dual of $S$ in $X$* to be

$$S^{*_X} := X \cap S^* = X \cap \mathrm{conv}(S)^*, \tag{3.39}$$

where $S^*$ denotes the convex dual of $S$ in $\mathbb{P}^N$.

Note that since $S$ is very compact, so is its (projective) convex hull. By the Tarski-Seidenberg Principle [BCR10, Section 1.4] it follows that $\mathrm{conv}(S)^*$ is semialgebraic, and therefore so is $S^{*_X}$. It is important to point out that the set $S^{*_X}$ depends on the choice of isomorphism $(\mathbb{P}^N)^\vee \cong \mathbb{P}^N$, and therefore on the choice of inner product on $\mathbb{R}^{N+1}$. In the following, we consider the standard inner product. We now prove some properties of this notion of convex duality, which follow directly from elementary properties of the usual convex hull and convex duality in projective space.

**Proposition 3.6.2.** *Let $S, T \subset X$ be connected, semialgebraic, and very compact sets.*

1. *$S^{*_X}$ is convex in $X$.*

2. *If $S \subset T$, then $T^{*_X} \subset S^{*_X}$.*

3. *$\mathrm{conv}_X(S) \subset (S^{*_X})^{*_X}$.*

**Example 3.6.3.** Unlike in the projective case, if $S$ is convex in $X$, the inclusion $S \subset (S^{*_X})^{*_X}$ can be strict. For example, in an affine chart on $\mathbb{P}^2$ let $X = \mathcal{V}(x^2 + y^2 - 1)$ be the unit circle and $S = \{(x, y) : x^2 + y^2 \leq 1, x \geq 0\} \cap X$ be a semicircle. Then, $S^* = \{(x, y) : x^2 + y^2 \leq 1, x \leq 0\} \cup \{(x, y) : x \geq 0, -1 \leq y \leq 1\}$, and $S^{*_X} = X$. Thus the inclusion $(S^{*_X})^{*_X} = X \supset S$ is strict. $\diamond$

**Lemma 3.6.4.** *The nonnegative Grassmannian is self-dual, that is,*

$$\mathrm{Gr}_{\geq 0}(k, n)^{*_{\mathrm{Gr}(k,n)}} = \mathrm{Gr}_{\geq 0}(k, n). \tag{3.40}$$

*Proof.* This follows from Proposition 3.5.1 with $n = k + m$ and $Z$ being the identity matrix, together with the fact that $C_{k,m,k+m}(Z)$ is the projective simplex, which is self-dual. $\square$

### 3.6.2 The extendable dual amplituhedron for $k = m = 2$

We now use the results from Subsection 3.6.1 to define a notion of dual amplituhedron.

**Definition 3.6.5.** We define the *extendable dual amplituhedron* to be

$$\widetilde{\mathcal{A}}_{k,m,n} := \mathcal{A}_{k,m,n}^{*\mathrm{Gr}(k,k+m)} = \mathrm{Gr}(k, k+m) \cap \mathrm{conv}(\mathcal{A}_{k,m,n})^* = \mathrm{Gr}(k, k+m) \cap C_{k,m,n}^*, \qquad (3.41)$$

where in the last equality we used Proposition 3.5.1.

Since $C_{k,m,n}$ is the convex hull of the $\binom{n}{k}$ points in (3.18), we have that

$$C_{k,m,n}^* = \left\{ Y \in \mathbb{P}(\bigwedge^k \mathbb{R}^{k+m}) \,:\, \langle Y\, i_1 \ldots i_k \rangle \geq 0 \,,\; \forall\, (i_1 \ldots i_k) \in \binom{[n]}{k} \right\}. \qquad (3.42)$$

Let us now focus on the case of $k = m = 2$. By eq. (3.28) and Theorem 3.5.7 we have

$$\begin{aligned}
\widetilde{\mathcal{A}}_{2,2,n}(Z) &= \mathrm{Gr}(2,4) \cap C_{2,2,n}(Z)^* = \mathrm{Gr}(2,4) \cap \widetilde{C}_{2,2,n}(W) \\
&= \mathrm{Gr}(2,4) \cap C_{2,2,n}(W) = \mathcal{A}_{2,2,n}(W) \,,
\end{aligned} \qquad (3.43)$$

where we specified the dependence of the amplituhedron and of the polytopes on the positive matrix. In particular, for $k = m = 2$ the extendable dual amplituhedron is again an amplituhedron. This also explains our notation for the dual amplituhedron.

We now argue that this notion of duality for the amplituhedron at $k = m = 2$ is natural from a physics point of view. For that, it is useful to think of the amplituhedron $\mathcal{A}_{k,m,n}$ as living in $\mathrm{Gr}(m, n)$, by identifying it with its image under the *twistor embedding* $\psi: \mathrm{Gr}(k, k+m) \to \mathrm{Gr}(m, n)$, see [Lam26, Section 2]. Let us denote the amplituhedron in the twistor embedding by $\mathcal{B}_{k,m,n} = \psi(\mathcal{A}_{k,m,n})$. By [Lam26, Theorem 4], we have that

$$\mathcal{B}_{k,2,n}(Z) = \overline{\Pi_{+,k}^\circ(Z) \cap \mathrm{Gr}(2, Z)} \,, \qquad (3.44)$$

where $\Pi_{+,k}^\circ(Z)$ is the connected component of the complement of the $n$ divisors $\langle ii+1 \rangle = 0$ in $\mathrm{Gr}(2, Z)$ on which the sequence $(\langle 12 \rangle, \langle 13 \rangle, \ldots, \langle 1n \rangle)$ has $k$ sign-flips; see [Lam26, Proposition 2]. In this notation, we identify the image of the extendable dual amplituhedron under the twistor embedding with

$$\psi(\widetilde{\mathcal{A}}_{2,2,n}(Z)) = \psi(\mathcal{A}_{2,2,n}(W)) = \mathcal{B}_{0,2,n}(Z) \,. \qquad (3.45)$$

From this we deduce that $\widetilde{\mathcal{A}}_{2,2,n}(Z)$ has the following semialgebraic description. It consists of all points $Y \in \mathrm{Gr}(2,4)$ such that

$$\begin{aligned}
&\langle Y\, ii+1 \rangle > 0 \quad \forall\, i = 1, \ldots, n-1, \quad \langle Y1n \rangle > 0 \,, \text{ and} \\
&\text{the sequence } (\langle Y\, 12 \rangle, \langle Y\, 13 \rangle \ldots, \langle Y\, 1n \rangle) \text{ has zero sign flips.}
\end{aligned} \qquad (3.46)$$

**Corollary 3.6.6** (Extendable dual amplituhedron for $k = m = 2$)**.** The extendable dual amplituhedron for $k = m = 2$ in twistor space is equal to the twisted amplituhedron, which has a semialgebraic description as in (3.46).

Therefore, for $k = m = 2$ the duality of Definition 3.6.5 corresponds physically to interchanging the maximal-helicity-violating (MHV) ($k = 2$) sector with the $\overline{\text{MHV}}$ sector; see also [Her+21, Section 2.3]. More generally, we are lead to the following question.

**Question 3.6.7.** Is the analogue of Corollary 3.6.6 also true for $m = 2$ and $k > 2$?

This chapter has given a new description of the amplituhedron in terms of exterior cyclic polytopes. It fits into a larger search for a good notion of a dual amplituhedron. By starting with duality of convex sets, we have given a candidate definition for what the underlying semialgebraic set of such a dual would look like, using our concrete polytopal description. However, we are far from a satisfying conclusion to the story, and we hope that our explorations will be useful for future efforts.

# Chapter 4

# Wondertopes

This chapter is based on the paper [Bra+25], which is joint work with Sarah Brauner, Chris Eur, and Raluca Vlad. The text remains largely unchanged from that paper, except that we have omitted some preliminary material and the appendix, which contains the combinatorial analogue of Theorem 4.1.4 in the context of matroids.

## 4.1   Introduction

Positive geometries were introduced by Arkani-Hamed–Bai–Lam [ABL17] as a mathematical framework that encodes integrands arising in the computation of scattering amplitudes. Various quantum field theories give rise to various positive geometries. We recall the technical definition of positive geometry from the Introduction.

**Definition 4.1.1.** A *positive geometry* is a pair $(X, X_{\geq 0})$ consisting of an $n$-dimensional irreducible complex normal projective variety $X$ defined over $\mathbb{R}$, and a semialgebraic subset $X_{\geq 0} \subset X(\mathbb{R})$, along with a unique nonzero meromorphic $n$-form $\Omega(X, X_{\geq}0)$, satisfying the following properties.

- The interior $X_{>0} := \mathrm{int}(X_{\geq 0})$ of $X_{\geq 0}$ is a smooth oriented real $n$-manifold, and the Euclidean closure of $X_{>0}$ is $X_{\geq 0}$.

- Let $\partial X$ be the Zariski closure of $X_{\geq 0} \setminus X_{>0}$, whose codimension 1 irreducible components are $C_1, \ldots, C_k$. Then, with $C_{i,\geq 0}$ as the Euclidean closure of $\mathrm{int}(C_i \cap (X_{\geq 0} \setminus X_{>0}))$ in $C_i(\mathbb{R})$, the $n$-form $\Omega(X, X_{\geq 0})$ satisfies the following recursive property:

  (a) if $n = 0$, then $X = X_{\geq 0}$ is a point and $\Omega(X, X_{\geq 0}) = \pm 1$, depending on orientation; and

  (b) if $n > 0$, the form $\Omega(X, X_{\geq 0})$ has simple poles along each $C_i$ and no other poles, and every $(C_i, C_{i,>0})$ is a positive geometry, with the orientation on $C_{i,>0}$ inherited from that of $X_{>0}$, such that the residue of $\Omega(X, X_{\geq 0})$ along $C_i$ satisfies

$$\mathrm{Res}_{C_i} \Omega(X, X_{\geq 0}) = \Omega(C_i, C_{i,\geq 0}).$$

The $n$-form $\Omega(X, X_{\geq 0})$ is called the *canonical form* of $(X, X_{\geq 0})$, and $X_{\geq 0}$ the *nonnegative part*. We call $(C_i, C_{i,\geq 0})$ a (codimension 1) *boundary component* of $(X, X_{\geq 0})$.

By definition, a positive geometry $(X, X_{\geq 0})$ defines a pair $(X, \partial X^1)$ of a variety and a divisor where $\partial X^1$ denotes the union of components of $\partial X$ of codimension 1 in $X$. Thus, one may naturally ask: (when) does a positive geometry $(X, X_{\geq 0})$ admit a version of log resolution of singularities? More precisely, we ask:

**Question 4.1.2.** For a positive geometry $(X, X_{\geq 0})$, (when) is there a positive geometry $(\widetilde{X}, \widetilde{X}_{\geq 0})$, along with a birational map $\pi : \widetilde{X} \to X$ restricting to a diffeomorphism $\widetilde{X}_{>0} \cong X_{>0}$, such that $\widetilde{X}$ is smooth and the boundary $\partial \widetilde{X}^1$ is a simple normal crossing divisor?

Such a pair $(\widetilde{X}, \widetilde{X}_{\geq 0})$ may not always exist; see Example 4.5.2. However, we affirmatively answer the question for a prototypical example of a positive geometry, namely, a polytope in a projective space. We achieve this by introducing a new family of positive geometries, whose nonnegative regions we call *wondertopes*. Let us now recall the construction of a wondertope from Chapter 1.

Let $V \cong \mathbb{R}^{n+1}$ be a real vector space. Let $\mathcal{P}$ be a full-dimensional (convex) polytope in $\mathbb{R}^n$, considered as a subset of the projective space $\mathbb{P}V$ by the identification $\mathbb{P}V \setminus \mathbb{P}H \cong \mathbb{R}^n$ for a hyperplane $H$ in $V$. We will abuse notation to write $\mathbb{P}V$ also for the complex projective space $\mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C})$, as we trust that the field will be clear from context; as per the types in Definition 4.1.1, the varieties considered throughout the paper are complex, while the semialgebraic sets live inside the real points of these varieties. The pair $(\mathbb{P}V, \mathcal{P})$ is a positive geometry [ABL17], whose canonical form is well-studied for its connection to adjoint hypersurfaces [KR20] and the volume of dual polytopes [ABL17, § 7.4].

A wondertope is constructed from a polytope $\mathcal{P} \subset \mathbb{P}V$ and a set $\mathcal{B}$ of proper linear subvarieties of $\mathbb{P}V$ satisfying the following properties:

(1) for every $F \in \mathcal{B}$, the intersection $F \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$,

(2) every hyperplane whose intersection with $\mathcal{P}$ is a facet of $\mathcal{P}$ is in $\mathcal{B}$, and

(3) $\mathcal{B}$ is a *building set* in the sense of [DP95, Theorem 2.3.(2)]; that is, for every intersection $L = \bigcap_{F \in \mathcal{F}} F$ of a subset $\mathcal{F}$ of $\mathcal{B}$, the subset

$$\mathcal{F}_L = \{F \in \mathcal{B} \mid F \text{ is minimal (by inclusion) among elements of } \mathcal{B} \text{ that contain } L\}$$

of $\mathcal{B}$ satisfies $\sum_{F \in \mathcal{F}_L} \mathrm{codim}_{\mathbb{P}V} F = \mathrm{codim}_{\mathbb{P}V} L$.

**Definition 4.1.3.** With notation as above, choose any ordering of $\mathcal{B} = \{F_1, \ldots, F_k\}$ such that $i \leq j$ if $F_i \subseteq F_j$, and define $X^{\mathcal{B}}$, also denoted $\mathbb{P}V^{\mathcal{B}}$, to be the sequential blow-up

$$X^{\mathcal{B}} := \mathrm{Bl}_{F_k}(\cdots (\mathrm{Bl}_{F_2}(\mathrm{Bl}_{F_1} \mathbb{P}V)) \cdots)$$

with the blow-down map $\pi_{\mathcal{B}} : X^{\mathcal{B}} \longrightarrow \mathbb{P}V$. Here, we repurpose notation to write $F_2$ for the strict transform of $F_2$ in $\mathrm{Bl}_{F_1} \mathbb{P}V$, and similarly for $F_3, \ldots, F_k$. The *wondertope* $\widetilde{\mathcal{P}}^{\mathcal{B}}$ is

$$\widetilde{\mathcal{P}}^{\mathcal{B}} = \text{the Euclidean closure of } \pi_{\mathcal{B}}^{-1}(\text{the interior of } \mathcal{P}) \text{ in } X^{\mathcal{B}}(\mathbb{R}).$$

That a wondertope is semialgebraic is verified in Corollary 4.2.6. Our main result in this chapter is the following.

**Theorem 4.1.4.** *The pair $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ is a positive geometry whose canonical form is the pullback $\pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P})$ of the canonical form of $(\mathbb{P}V, \mathcal{P})$. The boundary components of $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ are (the strict transforms of) the exceptional divisors $E_F$ for $F \in \mathcal{B}$ such that $\dim(\mathcal{P} \cap F) = \dim F$.*

The variety $X^{\mathcal{B}}$ is known as the *wonderful compactification* of the arrangement complement $C(\mathcal{B}) = \mathbb{P}V \setminus (\bigcup \mathcal{B})$, introduced by De Concini and Procesi in [DP95]. They showed that the boundary $X^{\mathcal{B}} \setminus C(\mathcal{B})$, which consists of (the strict transforms of) the exceptional divisors of our sequential blow-up, is a simple normal crossing divisor. Thus, Theorem 4.1.4 implies that the pair $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ is a log resolution of singularities of the positive geometry $(\mathbb{P}V, \mathcal{P})$ in the sense of Question 4.1.2.

For $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ to be a positive geometry, but not necessarily with simple normal crossing boundary components, the condition on $\mathcal{B}$ can be relaxed slightly; see Theorem 4.4.5. We caution however that removing any one of the three conditions on $\mathcal{B}$ above results in a failure of the conclusion of Theorem 4.1.4; see Examples 4.5.1, 4.5.6, and 4.5.7.

One notable special case of Theorem 4.1.4 arises from the *rank $(n-1)$ braid arrangement* $A_{n-1}$ comprised of hyperplanes $\{x_i - x_j = 0\}$ in $V = \mathbb{R}^n / \mathbb{R} \cdot (1, 1, \ldots, 1)$ for $1 \leq i < j \leq n$. The complement $C(A_{n-1}) = \mathbb{P}V \setminus A_{n-1}$ is isomorphic to $M_{0,n+1}$, the moduli space of $n + 1$ distinct points on $\mathbb{P}^1$. For an appropriate choice of building set, the corresponding wonderful compactification is isomorphic to the Deligne–Knudsen–Mumford compactification $\overline{M}_{0,n+1}$ [DM69]. The polytope in this context is a simplex, and the corresponding wondertope $(\overline{M}_{0,n+1})_{\geq 0}$ is combinatorially isomorphic to the $n$-associahedron, which parameterizes triangulations of an $(n + 1)$-gon—hence, it is sometimes called the "curvy associahedron," and was first studied by Devadoss [Dev99]. Note that this wondertope $(\overline{M}_{0,n+1})_{\geq 0}$ is the *tropical compactification* of $M_{0,n+1}$ in the sense of [AHL21b]. Theorem 4.1.4 then recovers the following result of Arkani-Hamed–He–Lam [AHL21a, Proposition 8.2], which uses work of Brown–Carr–Schneps [BCS10, Proposition 2.7]. See Section 4.5.1 for details.

**Corollary 4.1.5.** *The pair $(\overline{M}_{0,n+1}, (\overline{M}_{0,n+1})_{\geq 0})$ is a positive geometry with canonical form given by the Parke-Taylor form; see Equation 4.4.*

## Related works

In the study of bordifications of links of tropical moduli spaces [Bro25], wondertopes $\widetilde{\mathcal{P}}^{\mathcal{B}}$ of a polytope $\mathcal{P}$ with respect to an intersection-closed set $\mathcal{B}$ of linear subspaces appeared as

"blow-ups of polyhedral linear configurations." In a slightly different setting of stratified manifolds and real blow-ups[1] [Gai03], wondertopes essentially appeared as compactifications of interiors of polytopes. In a related direction, "poset associahedra" were introduced in [Gal24] as certain compactifications of the interior of Stanley's order polytope of a poset. Brown-Dupont [BD25] propose an interpretation of positive geometries using Deligne's mixed Hodge theory for complex varieties; in their setting, invariance under blow-ups is immediate, whereas our paper shows that the situation is more subtle in the recursive setting (see non-examples in Sections 4.5.2 and 4.5.3).

## Organization

The remainder of the chapter proceeds as follows. Section 4.2 discusses properties of semialgebraic subsets, and proves that wondertopes are indeed semialgebraic. In Section 4.3 we show that blowing-up a single face in a polytope results in a positive geometry. We then pair this result with induction in Section 4.4 to prove Theorem 4.1.4. Section 4.5 focuses on examples; we discuss $\overline{M}_{0,n}$ and Corollary 4.1.5 in Section 4.5.1, and feature several pathologies in Sections 4.5.2 and 4.5.3 that show the conditions imposed on $\mathcal{B}$ are necessary.

## 4.2 Semialgebraic subsets

We define and record some properties of semialgebraic subsets, and show that wondertopes are semialgebraic. Much of this section can be found in a standard reference in the general setting of *real spectra*, which is beyond our needs. For the convenience of the reader, we include here a self-contained account of semialgebraicity in a scheme over $\mathbb{R}$, and verify its compatibility with the notion of semialgebraicity in a projective space given in [ABL17]. The latter is the same definition used in Chapter 3; see 3.2.1. Readers willing to believe the semialgebraicity of a wondertope (Corollary 4.2.6) may skip this section.

**Definition 4.2.1.** A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ defines a subset $\{x \in \mathbb{R}^n : f(x) > 0\}$ of $\mathbb{R}^n$. The collection of *finite boolean combinations* of such subsets, i.e. the smallest collection that contains all such subsets and is closed under finite unions, finite intersections, and complements, is called the set of *semialgebraic subsets* of $\mathbb{R}^n$.

For example, a polytope and the interior of a polytope in $\mathbb{R}^n$ are semialgebraic. Semialgebraic subsets are known to satisfy the following properties, both of which follow from the Tarski–Seidenberg theorem.

**Proposition 4.2.2.** *[BCR10, Chapter 2]*

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[1]Loosely put, our notion of a *blow-up* in the sense used in algebraic geometry [Har77, Chapter II.7] replaces the blown-up locus with the projectivization of its normal bundle, whereas the "real blow-up" known as the "balls, beams, and plates construction" in [KT99] replaces the blown-up locus with the sphere-bundle of its normal bundle.

- *Images and preimages of semialgebraic subsets under an algebraic map are semialgebraic.*

- *The closure and the interior (with respect to the Euclidean topology on $\mathbb{R}^n$) of a semialgebraic subset are semialgebraic.*

We now define semialgebraic subsets of schemes. Let $A$ be a finitely generated $\mathbb{R}$-algebra, and $Y = \operatorname{Spec} A$ its affine scheme. Because the only automorphism of the field $\mathbb{R}$ is the identity, an $\mathbb{R}$-valued point $y \in Y(\mathbb{R})$, i.e. a ring map $y : A \to \mathbb{R}$, is uniquely determined by the maximal ideal $\mathfrak{m}_y = y^{-1}(0)$ of $A$. Thus, for the point $y$ in $\operatorname{Spec} A$ and an element $f \in A$, the value $f(y)$ is well-defined as the image of $f$ under $A/\mathfrak{m}_y \cong \mathbb{R}$. This allows us to define the following.

**Definition 4.2.3.** For the affine scheme $Y = \operatorname{Spec} A$ of a finitely generated $\mathbb{R}$-algebra $A$, a subset $S \subseteq Y(\mathbb{R})$ of its $\mathbb{R}$-valued points is *semialgebraic* if it is a finite boolean combination of subsets of the form $\{y \in Y(\mathbb{R}) \mid f(y) > 0\}$ for some $f \in A$. For a scheme $X$ of finite type over $\mathbb{R}$, a subset $S \subseteq X(\mathbb{R})$ is *semialgebraic* if $S \cap U$ is semialgebraic for each affine open subset $U \subset X$. By convention we assume that schemes are separated.

We observe two features of this definition:

- The definition of semialgebraic subsets in an affine scheme is compatible with the one for subsets of $\mathbb{R}^n$. More precisely, for the affine scheme $Y = \operatorname{Spec} A$, choosing a generating set $x_1, \ldots, x_n$ for the $\mathbb{R}$-algebra $A$ gives a closed embedding $\iota : Y \hookrightarrow \mathbb{A}^n(\mathbb{R})$. Then, a subset $S \subseteq Y(\mathbb{R})$ is semialgebraic if and only if $\iota(S) \subseteq \mathbb{R}^n$ is semialgebraic, since every $f \in A$ has a lift $\widetilde{f} \in \mathbb{R}[x_1, \ldots, x_n]$ such that any two such lifts restrict to the same function on $\iota(Y)(\mathbb{R})$.

- For a scheme $X$ of finite type over $\mathbb{R}$, it suffices to check the semialgebraicity condition for an open affine cover of $X$, since open subsets and finite unions of semialgebraic subsets are semialgebraic, and since finite type schemes and intersections of affine open subsets of (separated) schemes are quasi-compact.

Together, the two observations imply the following.

**Lemma 4.2.4.** *Let $\{(U_\alpha, \iota_\alpha)\}$ be an affine open cover of a scheme $X$ of finite type over $\mathbb{R}$, with closed embeddings $\iota_\alpha : U_\alpha \hookrightarrow \mathbb{R}^{n_\alpha}$. Then, a subset $S \subseteq X(\mathbb{R})$ is semialgebraic if $\iota_\alpha(U_\alpha \cap S)$ is semialgebraic for all $\alpha$.*

Proposition 4.2.2 generalizes straightforwardly to the setting of semialgebraic subsets of a scheme, as follows; we omit the proof.

**Proposition 4.2.5.** *Let $X$ and $Y$ be schemes of finite type over $\mathbb{R}$, and $\varphi : X \to Y$ a morphism (which is necessarily quasi-compact).*

- *Images and preimages of semialgebraic subsets are semialgebraic under $\varphi$.*

- *The closure and the interior (with respect to the Euclidean topology) of a semialgebraic subset are semialgebraic.*

**Corollary 4.2.6.** For any sequential blow-up $\pi : X^{\mathcal{B}} \to \mathbb{P}V$ of linear subspaces that do not intersect the interior of a polytope $\mathcal{P} \subset \mathbb{P}V$, the strict transform $\widetilde{\mathcal{P}}^{\mathcal{B}}$, i.e. the Euclidean closure of $\pi^{-1}(\mathrm{int}(\mathcal{P}))$ in $X^{\mathcal{B}}(\mathbb{R})$, is semialgebraic.

**Remark 4.2.7.** The authors of [ABL17] define a semialgebraic subset of a projective space as follows: A subset $S \subset \mathbb{P}^n(\mathbb{R})$ is said to be semialgebraic if it is the image under $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{R})$ of a semialgebraic subset $S \subseteq \mathbb{R}^{n+1}$, all of whose defining equalities and inequalities are given by homogeneous polynomials.

Let us verify that their definition agrees with ours. Proposition 4.2.5 implies that if $S \subset \mathbb{P}^n(\mathbb{R})$ is semialgebraic in their sense, then it is also in our sense. For the converse, since $\mathbb{P}^n$ is covered by the (finitely many) coordinate affine charts, it suffices to show the converse for a subset $S$ of $\mathbb{P}^n$ that avoids a coordinate hyperplane, say $\{x_0 = 0\}$. In that case, the converse holds because under $\mathbb{A}^n \cong U_0 = \mathbb{P}^n \setminus \{x_0 = 0\}$, we have

$$\{x \in \mathbb{A}^n(\mathbb{R}) : f(x) > 0\} =$$
$$\text{the image under } \mathbb{R}^{n+1} \setminus \{x_0 = 0\} \to U_0 \text{ of } \{(x_0, x) \in \mathbb{R}^{n+1} : F(x) > 0\}$$

for any polynomial $f$ and any even degree homogenization $F$ of $f$ by the variable $x_0$.

**Remark 4.2.8.** More concretely, for a building set $\mathcal{B}$ as in Section 1.2, consider the rational map

$$\varphi : \mathbb{P}V \dashrightarrow \mathbb{P}V \times \prod_{\mathbb{P}W \in \mathcal{B}} \mathbb{P}(V/W),$$

coming from the identity $\mathbb{P}V \to \mathbb{P}V$ on the first coordinate and the quotient $V \to V/W$ on the other coordinates. According to [DP95], the sequential blow-up $X^{\mathcal{B}}$ is isomorphic to the Zariski closure of the image of $\varphi$ inside $\mathbb{P}V \times \prod_{\mathbb{P}W \in \mathcal{B}} \mathbb{P}(V/W)$, with the blow-down map $\pi$ being projection onto the first factor $\mathbb{P}V$. Under this embedding, the interior of the wondertope $\widetilde{\mathcal{P}}^{\mathcal{B}}$ is cut out by the multi-homogeneous equations cutting out $X^{\mathcal{B}}$, together with the strict linear inequalities in the coordinates of the first factor $\mathbb{P}V$ corresponding to the facet hyperplanes of $\mathcal{P}$. Finding the multi-homogeneous equations and inequalitites that cut out the entire wondertope $\widetilde{\mathcal{P}}^{\mathcal{B}}$ inside this product $\mathbb{P}V \times \prod_{\mathbb{P}W \in \mathcal{B}} \mathbb{P}(V/W)$ would be an interesting future direction of study.

## 4.3 Blowing up a single face

As in the introduction, let $V$ be an $(n + 1)$-dimensional real vector space, and let $\mathcal{P}$ be a full-dimensional polytope in $\mathbb{P}V$. This defines a positive geometry $(\mathbb{P}V, \mathcal{P})$. The goal of this section is to prove the following "fundamental computation," from which the proof of Theorem 4.1.4 will follow via induction in Section 4.4.

**Theorem 4.3.1.** *Let $W \subset V$ be a proper subspace such that $\mathbb{P}W \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$. Let $\pi : X = \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V \to \mathbb{P}V$ be the blow-up. Let the* strict transform $\widetilde{\mathcal{P}}$ *of $\mathcal{P}$ be*

$$\widetilde{\mathcal{P}} = \text{the Euclidean closure of } \pi^{-1}(\text{the interior of } \mathcal{P}) \text{ in } X(\mathbb{R}).$$

*Then, the pair $(X, \widetilde{\mathcal{P}})$ is a positive geometry whose canonical form is $\pi^* \Omega(\mathbb{P}V, \mathcal{P})$.*

In Section 4.3.1, we prepare by collecting known results about polytopes, normal cones, and triangulations. Then, in Section 4.3.2, we use these to prove Theorem 4.3.1.

## 4.3.1 Polytopes, normal cones, and triangulations

A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$. Throughout, let $\mathcal{P}$ be a *(projective) polytope* in $\mathbb{P}V$, by which we mean a polytope in $\mathbb{P}V \setminus \mathbb{P}H \cong \mathbb{R}^n$ for some hyperplane $H$. Choosing $H^+$ to be one of the two open half-spaces in $V$ defined by $H$, we obtain the following polyhedral cone in $V$:

$$\widehat{\mathcal{P}} = \mathbb{R}_{\geq 0}\{v \in H^+ \mid \text{image of } v \text{ in } \mathbb{P}V \text{ lies in } \mathcal{P}\}$$
$$= \mathbb{R}_{\geq 0}\{v \in H^+ \mid \text{image of } v \text{ in } \mathbb{P}V \text{ is a vertex of } \mathcal{P} \subset (\mathbb{P}V \setminus \mathbb{P}H \cong \mathbb{A}^n(\mathbb{R}))\}.$$

This cone is *pointed* (or also called *strongly convex*), i.e. $\widehat{\mathcal{P}} \cap (-\widehat{\mathcal{P}}) = \{0\}$. Its image in $\mathbb{P}V$ recovers the original polytope $\mathcal{P}$. If $\mathcal{P} = \emptyset$, we set $\widehat{P} = \{0\}$. We will freely switch between considering a projective polytope as a polytope in $\mathbb{P}V \setminus \mathbb{P}H \cong \mathbb{R}^n$ for some hyperplane $H$ and as the image in $\mathbb{P}V$ of a pointed polyhedral cone in $V$. That is, we will switch between a projective polytope $\mathcal{P} \subset \mathbb{P}V$ and its corresponding pointed polyhedral cone $\widehat{\mathcal{P}} \subset V$.

**Definition 4.3.2.** For a linear subspace $W \subset V$ such that $\widehat{\mathcal{P}} \cap W$ is a face of $\widehat{\mathcal{P}}$, we define two pointed polyhedral cones $\widehat{\mathcal{P}_W} \subset W$ and $\widehat{\mathcal{P}^W} \subset V/W$ by

$$\widehat{\mathcal{P}_W} = \widehat{\mathcal{P}} \cap W \qquad \text{and} \qquad \widehat{\mathcal{P}^W} = \begin{cases} \{0\} & \text{if } \widehat{\mathcal{P}} \cap W = \{0\} \\ \text{the image of } \widehat{\mathcal{P}} \text{ under } V \to V/W & \text{otherwise.} \end{cases}$$

We call the polytope $\mathcal{P}_W \subset \mathbb{P}W$ the *face of $\mathcal{P}$ relative to $W$*, and call the polytope $\mathcal{P}^W \subset \mathbb{P}(V/W)$ the *normal polytope of $\mathcal{P}$ relative to $W$*.

The cone $\widehat{\mathcal{P}^W}$ is sometimes known as the *normal cone* of $\widehat{\mathcal{P}}$ relative to $W$. It is a pointed polyhedral cone (see for instance [Bro25, Lemma 4.7]), so that the normal polytope $\mathcal{P}^W$ is a well-defined projective polytope. We caution that our definition of normal polytope is not the same as the polar dual polytope constructed from the normal fan.

We observe that these induced polytopes $\mathcal{P}_W$ and $\mathcal{P}^W$ arise in our context via the following Proposition; for a proof, see [Bro25, Proposition 5.10].[2] For a description of blow-ups and exceptional divisors appearing below, see [Har77, p. II.7].

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[2][Bro25, Proposition 5.10] is stated for the more general case of a sequential blow-up of an intersection-closed collection of linear subspaces, which specializes easily to the case stated here.

**Proposition 4.3.3.** *Let $\pi : \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V \to \mathbb{P}V$ be the blow-up, and $E$ the exceptional divisor. Suppose $\mathcal{P}$ is full-dimensional. Then, under the canonical isomorphism $E \cong \mathbb{P}W \times \mathbb{P}(V/W)$, the strict transform $\widetilde{\mathcal{P}}$ satisfies $E \cap \widetilde{\mathcal{P}} \cong \mathcal{P}_W \times \mathcal{P}^W$. See Figure 4.1.*



Figure 4.1: Left: a cube in $\mathbb{P}^3$ and a $\mathbb{P}^1$ (green) intersecting the cube along an edge $F$. Right: the cube after blowing-up $\mathbb{P}^3$ along the $\mathbb{P}^1$, with $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $E_{\geq 0} \cong F \times \Delta^1$ (the latter in green). Here, $\Delta^1$ denotes the standard simplex of the projective line (see Proposition 4.3.6 (c) for a definition).

We now describe how faces and normal polytopes interact with triangulations.

**Definition 4.3.4.** A *triangulation* of $\mathcal{P}$ is a finite collection $\mathcal{T}$ of simplices in $\mathbb{P}V$ of dimensions all equal to $\dim \mathcal{P}$ such that

  (i) the simplices in $\mathcal{T}$ cover $\mathcal{P}$, i.e. $\bigcup_{T \in \mathcal{T}} T = \mathcal{P}$, and

  (ii) any two simplices $T, T' \in \mathcal{T}$ intersect in a (possibly empty) common face.

Let us fix a triangulation $\mathcal{T}$ of $\mathcal{P}$, and fix a linear subspace $W \subset V$ such that $\mathbb{P}W$ intersects $\mathcal{P}$ along a (possibly empty) face. Note that $\mathbb{P}W$ then intersects each simplex $T \in \mathcal{T}$ along a (possibly empty) face, so we may consider $T_W$ and $T^W$.

**Lemma 4.3.5.** *One has triangulations of $\mathcal{P}_W$ and $\mathcal{P}^W$ obtained from $\mathcal{T}$ as follows.*

  (a) *The collection $\mathcal{T}_W = \{T_W \mid T \in \mathcal{T}$ with $\dim T_W = \dim \mathcal{P}_W\}$ is a triangulation of $\mathcal{P}_W$.*

  (b) *Suppose further that $\dim \mathcal{P}_W = \dim \mathbb{P}W$. Then, for any simplex $F$ in the triangulation $\mathcal{T}_W$ of $\mathcal{P}_W$, the collection $\mathcal{T}^{W,F} = \{T^W \mid T \in \mathcal{T}$ with $T_W = F\}$ is a triangulation of $\mathcal{P}^W$.*

*Proof.* We omit the proof of the first statement (a), as it is straightforward. We prove the second statement (b), which is false without the assumption $\dim \mathcal{P}_W = \dim \mathbb{P}W$.

Every $T \in \mathcal{T}^{W,F}$ is a full-dimensional simplex in $\mathbb{P}V$ which intersects $\mathbb{P}W$ along $F$, a full-dimensional simplex in $\mathbb{P}W$. Hence the normal face $T^W$ of $T$ relative to $\mathbb{P}W$ is a full-dimensional simplex in $\mathbb{P}(V/W)$. With the cone $\widehat{\mathcal{P}}$ obtained by a choice of open half-space $H^+ \subset V$, we let $\widehat{T}$ be the cone of a simplex $T \in \mathcal{T}$ for the same half-space $H^+$. We need to show that $\{\widehat{T^W} \mid T^W \in \mathcal{T}^{W,F}\}$ is a triangulation of $\widehat{\mathcal{P}^W}$.

Pick any non-zero $v \in \widehat{\mathcal{P}} \subset V$ with image $\overline{v} \in \widehat{\mathcal{P}^W} \subset V/W$. We will show that $\overline{v}$ belongs to $\widehat{T^W}$ for some $T^W \in \mathcal{T}^{W,F}$. Consider a ray $w \in \widehat{\mathcal{P}}$ whose projectivization lies in the interior of $F$. Then for any positive integer $c$, both $\frac{1}{c} \cdot v$ and $w$ lie in $\widehat{\mathcal{P}}$, so the sum $\frac{1}{c} \cdot v + w$ does as well. Since our original triangulation $\mathcal{T}$ covers $\mathcal{P}$, there must exist a $T \in \mathcal{T}$ whose cone $\widehat{T}$ contains $\frac{1}{c} \cdot v + w$ for infinitely many positive integers $c$. This cone $\widehat{T} \subset V$ is closed in the real topology induced by any linear identification $V \cong \mathbb{R}^{n+1}$, so $\widehat{T}$ must also contain the limit point $w$. By assumption, $w$ lies in the interior of $F$, so $T^W \in \mathcal{T}^{W,F}$ by part (a). Its normal cone $\widehat{T^W}$ thus contains $\overline{v + c \cdot w} = \overline{v}$.

Finally, consider any simplices $T_1, T_2 \in \mathcal{T}^{W,F}$. Since $\mathcal{T}$ is a triangulation, $T_1$ intersects $T_2$ in a common face $S = T_1 \cap T_2$. We claim that $T_1^W \cap T_2^W = S^W$, which is a common face. By construction, both $T_1$ and $T_2$ intersect $\mathbb{P}W$ along $F$, which is a full-dimensional simplex in $\mathbb{P}W$. Thus the same must be true for $S = T_1 \cap T_2$. It follows that $\widehat{S^W} \subseteq \widehat{T_1^W} \cap \widehat{T_2^W}$ is a common face. In fact, this containment is an equality since, by assumption, the cones $\widehat{T_1}$ and $\widehat{T_2}$ intersect only along $\widehat{S}$. $\qquad\square$

In our context, the utility of considering triangulations arises from the following known results in positive geometry.

**Proposition 4.3.6.** *For a full-dimensional polytope $\mathcal{P} \subset \mathbb{P}V$, choose an orientation of its interior by an orientation of an affine chart $\mathbb{A}^n(\mathbb{R}) \cong \mathbb{R}^n$ containing it.*

(a) *[ABL17, Section 6.1] The pair $(\mathbb{P}V, \mathcal{P})$ is a positive geometry.*

(b) *[ABL17, Section 3.1] If $\mathcal{T}$ is a triangulation of $\mathcal{P}$, all of whose simplices are oriented the same way as $\mathcal{P}$, then the canonical forms of $(\mathbb{P}V, \mathcal{P})$ and $(\mathbb{P}V, T)$ for $T \in \mathcal{T}$ satisfy*

$$\Omega(\mathbb{P}V, \mathcal{P}) = \sum_{T \in \mathcal{T}} \Omega(\mathbb{P}V, T).$$

(c) *[ABL17, Section 5.1] Suppose $\mathcal{P}$ is the standard simplex $\Delta^n = \{[X_0 : \ldots : X_n] \mid X_i \geq 0 \text{ for all } i\} \subset \mathbb{P}^n(\mathbb{R})$. Let $U$ be the affine chart $\mathbb{A}^n \cong U = \{X_0 \neq 0\} \subset \mathbb{P}^n$ with coordinates $x_1 = \frac{X_1}{X_0}, \ldots, x_n = \frac{X_n}{X_0}$, in which the interior of $\Delta^n$ is identified with the positive orthant $\{x \in \mathbb{A}^n(\mathbb{R}) \mid x_i > 0 \text{ for all } i\}$. If the standard orientation of $\mathbb{A}^n(\mathbb{R}) \cong U(\mathbb{R})$ given by the ordered coordinates $(x_1, \ldots, x_n)$ agrees with the orientation of $\mathrm{int}(\Delta^n)$, then the canonical form satisfies*

$$\Omega(\mathbb{P}^n, \Delta^n)|_U = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

Lastly, we record the orientation and residue conventions from [ABL17, Appendix A] in the following remark, with a focus on the standard simplex $\Delta^n$ from Proposition 4.3.6(c).

**Remark 4.3.7.** Given a boundary component $(C, C_{\geq 0})$ of $(X, X_{\geq 0})$, pick a (boundary) chart $U \subset X_{\geq 0}$ diffeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that the restriction $U \cap C_{>0}$ is identified with $\mathbb{R}^{n-1} \times \{0\}$. For instance, one can pick $U$ to be an analytic neighborhood around a point of $C_{>0}$. We may choose the identification $U \cong \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that the restricted identification between $U \cap X_{>0}$ and $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ is orientation preserving, where $\mathbb{R}^{n-1} \times \mathbb{R}_{>0} \subset \mathbb{R}^n$ has the standard orientation. We orient $C_{>0}$ such that the identification between $U \cap C_{>0}$ and $\mathbb{R}^{n-1} \times \{0\}$ is orientation preserving, where $\mathbb{R}^{n-1} \times \{0\}$ has the standard orientation. We say that this is the orientation that $X_{>0}$ *induces on* $C_{>0}$.

For the residue, if a divisor is locally given by an equation $f = 0$, then the residue of a form with a simple pole at $\{f = 0\}$, when locally written as $\alpha \wedge \frac{df}{f}$, is $\alpha|_{f=0}$. Let us explicitly describe these conventions in the case of the standard simplex $\Delta^n \subset \mathbb{P}^n(\mathbb{R})$.

Consider the boundary component $(\{X_n = 0\}, \Delta^n \cap \{X_n = 0\}) \cong (\mathbb{P}^{n-1}, \Delta^{n-1})$. It intersects our chart $\mathbb{A}^n$ along $\mathbb{A}^{n-1} \cong \{x_n = 0\} \subset \mathbb{A}^n$. Since the standard Euclidean orientation on $\mathbb{A}^n(\mathbb{R})$ induces the standard Euclidean orientation on the hyperplane $\mathbb{A}^{n-1}(\mathbb{R}) \cong \{x_n = 0\}$, then we consider $\Delta^{n-1}$ oriented by the standard orientation on the real points of our chart $\mathbb{A}^{n-1} \cong \{X_0 \neq 0\} \subset \mathbb{P}^{n-1}$. And indeed, the recursive formula in Definition 4.1.1 holds:

$$\mathrm{Res}_{\{x_n=0\}} \; \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-1}}{x_{n-1}}.$$

However, consider instead $(\{X_{n-1} = 0\}, \Delta^n \cap \{X_{n-1} = 0\}) \cong (\mathbb{P}^{n-1}, \Delta^{n-1})$. Because the standard Euclidean orientation on $\mathbb{A}^n(\mathbb{R})$ induces the opposite orientation on $\mathbb{A}^{n-1}(\mathbb{R}) \cong \{x_{n-1} = 0\}$, we consider $\Delta^{n-1}$ oriented by the opposite orientation on our chart $\mathbb{A}^{n-1} \cong \{X_0 \neq 0\} \subset \mathbb{P}^{n-1}$. The recursive formula in Definition 4.1.1 catches this opposite orientation:

$$\mathrm{Res}_{\{x_{n-1}=0\}} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} = \mathrm{Res}_{\{x_{n-1}=0\}} \left( -\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-2}}{x_{n-2}} \wedge \frac{dx_n}{x_n} \right) \wedge \frac{dx_{n-1}}{x_{n-1}}$$

$$= -\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-2}}{x_{n-2}} \wedge \frac{dx_n}{x_n}.$$

## 4.3.2 The fundamental computation

Let us recall the setting of Theorem 4.3.1. That is, let

- $\mathcal{P}$ be a full-dimensional (oriented) polytope in $\mathbb{P}V$,

- $W \subset V$ be a proper linear subspace such that $\mathbb{P}W \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$,

- $\pi : X = \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V \to \mathbb{P}V$ be the blow-up, with exceptional divisor $E$, and

- $\widetilde{\mathcal{P}} \subset X(\mathbb{R})$ be the strict transform of $\mathcal{P}$, i.e. the Euclidean closure of $\pi^{-1}(\mathrm{int}(\mathcal{P}))$.

Since $\pi$ restricts to an isomorphism $X \setminus E \xrightarrow{\sim} \mathbb{P}V \setminus \mathbb{P}W$, in particular a diffeomorphism $\pi^{-1}(\mathrm{int}(\mathcal{P})) \to \mathrm{int}(\mathcal{P})$, the interior $\mathrm{int}(\widetilde{\mathcal{P}})$ of $\widetilde{\mathcal{P}}$ is a real $n$-manifold, which we orient such that $\pi$ is orientation preserving. Let $\partial\widetilde{\mathcal{P}}$ be the Zariski closure of $\widetilde{\mathcal{P}} \setminus \mathrm{int}(\widetilde{\mathcal{P}})$ inside $X$. The following two propositions constitute the key computations for the proof of Theorem 4.3.1.

**Proposition 4.3.8.** *The exceptional divisor $E$ is an irreducible component of $\partial\widetilde{\mathcal{P}}$ if and only if $\dim\mathcal{P}_W = \dim\mathbb{P}W$. When $\dim\mathcal{P}_W < \dim\mathbb{P}W$, the form $\pi^*\Omega(\mathbb{P}V,\mathcal{P})$ has no pole along $E$.*

Define $E_{\geq 0} = E \cap \widetilde{\mathcal{P}}$. Under the isomorphism $E \cong \mathbb{P}W \times \mathbb{P}(V/W)$, recall that $E_{\geq 0} \cong \mathcal{P}_W \times \mathcal{P}^W$ (Proposition 4.3.3).

**Proposition 4.3.9.** *When $\dim\mathcal{P}_W = \dim\mathbb{P}W$, the form $\pi^*\Omega(\mathbb{P}V,\mathcal{P})$ has a simple pole along $E$, and under the isomorphism $E \cong \mathbb{P}V \times \mathbb{P}(V/W)$, its residue along $E$ satisfies*

$$\mathrm{Res}_E(\pi^*\Omega(\mathbb{P}V,\mathcal{P})) = \Omega(\mathbb{P}W,\mathcal{P}_W) \wedge \Omega(\mathbb{P}(V/W),\mathcal{P}^W).$$

*Moreover, the pair $(E, E_{\geq 0})$ is a positive geometry with the canonical form $\mathrm{Res}_E(\pi^*\Omega(\mathbb{P}V,\mathcal{P}))$.*

We record here the following lemma, which, though standard, will be of use in proving Theorem 4.3.1, Lemma 4.3.12 and Theorem 4.4.5.

**Lemma 4.3.10.** *For a smooth irreducible complex projective variety $X$ of dimension $n$, if two meromorphic $n$-forms on $X$ agree on a Zariski dense open subset, then they agree on $X$.*

*Proof.* As $X$ is projective, a meromorphic $n$-form on $X$ is a rational section of the canonical bundle $\omega_X$ [GH94, pg. 170]. The lemma thus follows from the general fact that if two rational sections of a torsion-free sheaf on an integral scheme agree on a Zariski open subset, then they are equal. $\qquad\square$

We now prove Theorem 4.3.1, reproduced below, postponing the proofs of Propositions 4.3.8 and 4.3.9.

**Theorem 4.3.1.** *Let $W \subset V$ be a proper subspace such that $\mathbb{P}W \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$. Let $\pi : X = \mathrm{Bl}_{\mathbb{P}W}\,\mathbb{P}V \to \mathbb{P}V$ be the blow-up. Let the strict transform $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ be*

$$\widetilde{\mathcal{P}} = \text{the Euclidean closure of } \pi^{-1}(\text{the interior of } \mathcal{P}) \text{ in } X(\mathbb{R}).$$

*Then, the pair $(X, \widetilde{\mathcal{P}})$ is a positive geometry whose canonical form is $\pi^*\Omega(\mathbb{P}V,\mathcal{P})$.*

*Proof.* We induct on $\dim\mathbb{P}V$. The base case $\dim\mathbb{P}V = 1$ holds trivially since $\mathbb{P}W \subset \mathbb{P}V$ has codimension at most 1 so that the blow-up is an isomorphism on all of $\mathbb{P}V$.

For the general case, without loss of generality, suppose the codimension of $\mathbb{P}W \subset \mathbb{P}V$ is at least 2. We must verify that the residues of $\pi^*\Omega(\mathbb{P}V,\mathcal{P})$ are the canonical forms of the boundary components of $\partial\widetilde{\mathcal{P}}$. The boundary components fall into two possible cases: (i) the strict transform $\widetilde{\mathbb{P}H}$ of a hyperplane $\mathbb{P}H \subset \mathbb{P}V$ such that $\mathcal{P}_H = \mathbb{P}H \cap \mathcal{P}$ is a facet of $\mathcal{P}$, and (ii) the exceptional divisor $E$. These are the only possible boundary components because the boundary components of $\partial\mathcal{P}$ are its facet hyperplanes, $\pi$ is an isomorphism outside of $E$, and $\pi(\widetilde{\mathcal{P}}) = \mathcal{P}$.

In the case of (i), by the universal property of blow-ups, the strict transform $\widetilde{\mathbb{P}H}$ is the blow-up $\mathrm{Bl}_{\mathbb{P}W\cap\mathbb{P}H}\,\mathbb{P}H$, whose exceptional divisor $E_H$ is $E \cap \widetilde{\mathbb{P}H}$, and whose blow-down map

$\pi_H : \widetilde{\mathbb{P}H} \to \mathbb{P}H$ is the restriction $\pi|_{\widetilde{\mathbb{P}H}}$. Let $\widetilde{\mathcal{P}_H}$ be the strict transform of $\mathcal{P}_H$ under $\pi_H$, i.e. the Euclidean closure of $\pi_H^{-1}(\text{int}(\mathcal{P}_H))$ in $\widetilde{\mathbb{P}H}(\mathbb{R})$. By the induction hypothesis, the pair $(\widetilde{\mathbb{P}H}, \widetilde{\mathcal{P}_H})$ is a positive geometry.

To see that $(\widetilde{\mathbb{P}H}, \widetilde{\mathcal{P}_H})$ is a boundary component of $(X, \widetilde{\mathcal{P}})$ as a positive geometry, we first note that $\widetilde{\mathcal{P}_H}$ is the Euclidean closure of the interior of $\widetilde{\mathcal{P}} \cap \widetilde{\mathbb{P}H}$ in $\widetilde{\mathbb{P}H}(\mathbb{R})$. Indeed, outside of the codimension 1 closed locus $E_H$, the map $\pi$ identifies $(\widetilde{\mathcal{P}} \cap \widetilde{\mathbb{P}H}) \setminus E_H$ with $\mathcal{P}_H \setminus \mathbb{P}W$, so that $\widetilde{\mathcal{P}} \cap \widetilde{\mathbb{P}H}$ and $\widetilde{\mathcal{P}_H}$ share the interior $\pi_H^{-1}(\text{int}(\mathcal{P}_H))$. Finally, to verify the appropriate residue property, note that the Zariski open subset $X \setminus E$ of $X$ intersects $\widetilde{\mathbb{P}H}$ in a Zariski open subset, and $\pi$ is an isomorphism on $X \setminus E$. Hence, when restricted to $X \setminus E$, we find

$$\text{Res}_{\widetilde{\mathbb{P}H}}(\pi^*\Omega(\mathbb{P}V, \mathcal{P})) = \pi_H^*\big( \text{Res}_{\mathbb{P}H} \Omega(\mathbb{P}V, \mathcal{P})\big) = \pi_H^*\big(\Omega(\mathbb{P}H, \mathcal{P}_H)\big) = \Omega(\widetilde{\mathbb{P}H}, \widetilde{\mathcal{P}_H}).$$

The first equality follows because taking residues commutes with pulling back by isomorphisms. The second equality follows because $(\mathbb{P}V, \mathcal{P})$ is a positive geometry with $(\mathbb{P}H, \mathcal{P}_H)$ as a boundary component. When we restrict to the Zariski open subset $X \setminus E$ we obtain an equality of rational forms $\text{Res}_{\widetilde{\mathbb{P}H}}(\pi^*\Omega(\mathbb{P}V, \mathcal{P})) = \Omega(\widetilde{\mathbb{P}H}, \widetilde{\mathcal{P}_H})$. This extends to an equality on all of $X$ (see Lemma 4.3.10).

In the case of (ii), if $\dim \mathcal{P}_W < \dim \mathbb{P}W$, then by Proposition 4.3.8 the exceptional divisor $E$ is not a boundary component, and $\pi^*\Omega(\mathbb{P}V, \mathcal{P})$ has no pole along $E$. If $\dim \mathcal{P}_W = \dim \mathbb{P}W$, then by Proposition 4.3.9, the exceptional divisor $E$ is a boundary component with $E_{\geq 0} = E \cap \widetilde{\mathcal{P}}$, whose canonical form is the residue $\text{Res}_E(\pi^*\Omega(\mathbb{P}V, \mathcal{P}))$.

We have shown that $\pi^*\Omega(\mathbb{P}V, \mathcal{P})$ satisfies the recursive property in Definition 4.1.1. Now, suppose $\Omega'$ is another form with the same recursive property. Then the difference $\pi^*\Omega(\mathbb{P}V, \mathcal{P}) - \Omega'$ has no poles and is thus holomorphic on $X$. This difference is necessarily zero: $X$ has no nonzero holomorphic $n$-forms because $X$ is rational and the geometric genus is a birational invariant of smooth projective varieties [Har77, p. II.8.19]. $\qquad\square$

The rest of this section proves Propositions 4.3.8 and 4.3.9. For both propositions, we may assume that the codimension of $\mathbb{P}W \subset \mathbb{P}V$ is at least 2, since otherwise the statements hold trivially.

*Proof of Proposition 4.3.8.* Recall that $\widetilde{\mathcal{P}} \cap E \cong \mathcal{P}_W \times \mathcal{P}^W \subset \mathbb{P}W \times \mathbb{P}(V/W)$ by Proposition 4.3.3. Since $\dim \mathcal{P}^W = \dim \mathbb{P}(V/W)$ as long as $\mathbb{P}W \cap \mathcal{P} \neq \emptyset$, we find that $\widetilde{\mathcal{P}} \cap E$ is full-dimensional in $E$ if and only if $\dim \mathcal{P}_W = \dim \mathbb{P}W$. This proves the first statement.

For the second statement, suppose $\dim \mathcal{P}_W < \dim \mathbb{P}W$, and let us fix a triangulation $\mathcal{T}$ of $\mathcal{P}$, so that Proposition 4.3.6(b) gives us $\Omega(\mathbb{P}V, \mathcal{P}) = \sum_{T \in \mathcal{T}} \Omega(\mathbb{P}V, T)$. We claim that $\pi^*\Omega(\mathbb{P}V, T)$ has no pole along $E$ for every $T \in \mathcal{T}$, which implies that $\pi^*(\mathbb{P}V, \mathcal{P})$ has no pole along $E$. For proof of the claim, let $\partial T$ be the collection of $\dim \mathbb{P}V + 1$ many facet hyperplanes of a simplex $T \in \mathcal{T}$. Since $\dim(T \cap \mathbb{P}W) < \dim \mathbb{P}W$, we find that

$$\text{(the number of hyperplanes in } \partial T \text{ containing } \mathbb{P}W) < \text{codim}_{\mathbb{P}V} \mathbb{P}W.$$

The claim now follows from the following general Lemma. $\qquad\square$

**Lemma 4.3.11.** *Let $Y$ be a smooth irreducible closed subvariety of codimension at least $2$ in a smooth complex variety $X$ of dimension $n$. Let $\pi : X' = \mathrm{Bl}_Y X \to X$ be the blow-up with exceptional divisor $E$. For a collection $D$ of (distinct) smooth irreducible divisors $D_1, \ldots, D_m$ in $X$, let*

$$\mathrm{mult}_Y D = \text{the number of } D_i \text{ containing } Y.$$

*If $\mathrm{mult}_Y D < \mathrm{codim}_X Y$, then for any meromorphic $n$-form $\Omega$ on $X$ with at most simple poles along $D_1, \ldots, D_m$, the pullback $\pi^* \Omega$ has no pole along $E$.*

*Proof.* Since $\pi$ restricts to an isomorphism on $\pi^{-1}(X \setminus Y)$, it suffices to verify that, for any point $y \in Y$ and an analytic neighborhood $U \ni y$ in $X$, the restriction $\pi^* \Omega|_{\pi^{-1}(U)}$ has no pole along $E$. Since $Y$ and $D_1, \ldots, D_m$ are smooth, shrinking $U$ if necessary, we may choose coordinates $x_1, \ldots, x_n$ of $U$ to have $U \subseteq \mathbb{A}^n$ and $y = (0, \ldots, 0)$ such that $Y|_U$ is the subspace $L = \{x_{k+1} = \cdots = x_n = 0\}$ and each $D_i|_U$ is either empty or is the hyperplane $\{H_i = 0\}$ for some linear function $H_i$. Suppose $D_1|_U, \ldots, D_\ell|_U$ are nonempty. By the assumption that $\mathrm{mult}_Y D < \mathrm{codim}_X Y = n - k$, at most $n - k - 1$ of the functions $H_1, \ldots, H_\ell$ satisfy $H_i|_L \equiv 0$.

We now carry out the blow-up computation. We have

$$\mathrm{Bl}_L U = \left\{ \big((x_1, \ldots, x_n), [Y_{k+1} : \cdots : Y_n]\big) \mid x_i Y_j = x_j Y_i \text{ for all } i, j \right\} \subset U \times \mathbb{P}^{n-k-1}$$

and $\pi$ is just projection onto the first factor. We work in the affine open chart

$$U' = \{Y_n \neq 0\} \subset \mathrm{Bl}_L U,$$

where $U'$ is considered as an open subset of $\mathbb{A}^n$ via local coordinates $x_1, \ldots, x_k, y_{k+1}, \ldots, y_{n-1}, x_n$ given by $y_i = \frac{Y_i}{Y_n}$. A point

$$(x_1, \ldots, x_k, y_{k+1}, \ldots, y_{n-1}, x_n) \in U'$$

corresponds to

$$\big((x_1, \ldots, x_k, y_{k+1} x_n, \ldots, y_{n-1} x_n, x_n), [y_{k+1} : \cdots : y_{n-1} : 1]\big) \in \mathrm{Bl}_L U.$$

In this chart, the blow-down map $\pi : U' \to U$ sends

$$(x_1, \ldots, x_k, y_{k+1}, \ldots, y_{n-1}, x_n) \mapsto (x_1, \ldots, x_k, y_{k+1} x_n, \ldots, y_{n-1} x_n, x_n). \tag{4.1}$$

Since $\Omega$ has at most simple poles along the $D_i$, we have $\Omega|_U = \frac{f(x)}{H_1 \cdots H_\ell} dx_1 \cdots dx_n$ for a holomorphic function $f$ on $U \subseteq \mathbb{A}^n$. Writing $g(x, y)$ for the function $f(x_1, \ldots, x_k, y_{k+1} x_n, \ldots, y_{n-1} x_n, x_n)$ on $U'$, we have

$$\begin{aligned}
\big(\pi^* \Omega\big)\big|_{U'} &= \frac{g(x, y) \, dx_1 \wedge \cdots \wedge dx_k \wedge d(y_{k+1} x_n) \wedge \cdots \wedge d(y_{n-1} x_n) \wedge dx_n}{\prod_{i=1}^{\ell} H_i(x_1, \ldots, x_k, y_{k+1} x_n, \ldots, y_{n-1} x_n, x_n)} \\
&= \frac{g(x, y) \cdot x_n^{n-k-1} \, dx_1 \wedge \cdots \wedge dx_k \wedge dy_{k+1} \wedge \cdots \wedge dy_{n-1} \wedge dx_n}{\prod_{i=1}^{\ell} H_i(x_1, \ldots, x_k, y_{k+1} x_n, \ldots, y_{n-1} x_n, x_n)}.
\end{aligned}$$

Since the exceptional divisor $E$ on our chart $U'$ is given by $\{x_n = 0\}$, we now verify that $\left(\pi^* \Omega\right)\big|_{U'}$ has no poles along $\{x_n = 0\}$. As at most $n-k-1$ many of the $H_i$ satisfy $H_i|_L \equiv 0$, we find that at most $n-k-1$ many of the $H_i$ satisfy

$$x_n \text{ divides } H_i(x_1, \ldots, x_k, y_{k+1}x_n, \ldots, y_{n-1}x_n, x_n).$$

Therefore, we have at most $n-k-1$ factors of $x_n$ in the denominator of $\left(\pi^* \Omega\right)\big|_{U'}$ above, which are canceled by $x_n^{n-k-1}$ in the numerator. $\qquad\square$

For the proof of Proposition 4.3.9, we prepare by first considering the case when $\mathcal{P}$ is a simplex. One may deduce Lemma 4.3.12 by using the fact that projective normal toric varieties are positive geometries [ABL17, Section 3.6], but we provide an elementary proof.

**Lemma 4.3.12.** *Proposition 4.3.9 holds when $\mathcal{P}$ is a simplex.*

*Proof.* By a linear change of coordinates, we can assume that $\mathcal{P}$ is the standard simplex $\Delta^n \subset \mathbb{P}^n(\mathbb{R})$, and that the linear subspace $\mathbb{P}W$ we are blowing-up is the subspace $\{X_{k+1} = \cdots = X_n = 0\} \subset \mathbb{P}^n$. Note that both the face $\mathcal{P}_W \subset \mathbb{P}W$ and the normal polytope $\mathcal{P}^W \subset \mathbb{P}(V/W)$ are the standard simplices in $\mathbb{P}^k$ and $\mathbb{P}^{n-k-1}$, respectively. Performing a linear change of coordinates if necessary, we may moreover assume that the orientation of the interior of $\mathcal{P} = \Delta^n \subset \mathbb{P}^n(\mathbb{R})$ is compatible with the standard orientation on the real points of the affine chart $\mathbb{A}^n \cong U = \{X_0 \neq 0\} \subset \mathbb{P}^n$ with coordinates $x_i = \frac{X_i}{X_0}$.

Let $\Omega(\Delta^n)$ be the canonical form of $(\mathbb{P}^n, \Delta^n)$. Under the blow-up $\pi : \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}^n \to \mathbb{P}^n$, and the identification $E \cong \mathbb{P}^k \times \mathbb{P}^{n-k-1}$, we show that $\pi^* \Omega(\Delta^n)$ has a simple pole along $E$, and that $\mathrm{Res}_E \pi^* \Omega(\Delta^n) = \Omega(\Delta^k) \wedge \Omega(\Delta^{n-k-1})$. By Lemma 4.3.10, it suffices to show these on a Zariski dense open subset $U'$ of $\mathrm{Bl}_{\mathbb{P}W} \mathbb{P}^n$ such that $U' \cap E$ is nonempty (and hence dense in $E$). To this end, consider the affine chart $\mathbb{A}^n \cong U = \{X_0 \neq 0\} \subset \mathbb{P}^n$, over which we have $\mathrm{Bl}_L \mathbb{A}^n \cong \pi^{-1}(U)$ where $L$ is the subspace $\{x_{k+1} = \ldots = x_n = 0\}$. Recalling that

$$\mathrm{Bl}_L \mathbb{A}^n = \left\{ \big((x_1, \ldots, x_n), [Y_{k+1} : \cdots : Y_n]\big) \mid x_i Y_j = x_j Y_i \text{ for all } i, j \right\} \subset \mathbb{A}^n \times \mathbb{P}^{n-k-1},$$

we consider the affine open chart $U' = \{Y_n \neq 0\}$ in $\mathrm{Bl}_L \mathbb{A}^n$. As in the proof of Lemma 4.3.11, let $x_1, \ldots, x_k, y_{k+1}, \ldots, y_{n-1}, x_n$ be local coordinates on $U'$ given by $y_i = \frac{Y_i}{Y_n}$. The exceptional divisor $E$ is given by $\{x_n = 0\}$ in this chart, and the blow-down map $\pi : \mathbb{A}^n \cong U' \to U \cong \mathbb{A}^n$ is as in (4.1).

The determinant of the Jacobian matrix of this map is a power of $x_n$, and is thus positive when $x_n > 0$. Consider the standard orientation on $U'(\mathbb{R})$, given by the ordered coordinates $(x_1, \ldots, x_k, y_{k+1}, \ldots, y_{n-1}, x_n)$. Since the points in $\mathrm{int}(\mathcal{P})$ and $\pi^{-1}(\mathrm{int}(\mathcal{P}))$ satisfy $x_n > 0$, this orientation is compatible via $\pi$ to the standard orientation on $U(\mathbb{R})$ that we picked, which was given by the ordered coordinates $(x_1, \ldots, x_n)$.

By Proposition 4.3.6(c), the canonical form $\Omega = \Omega(\mathbb{P}^n, \Delta^n)$ is given by $\Omega|_U = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$. Thus, the pullback $\pi^*\Omega$ in the local coordinates on $U'$ is

$$
\begin{aligned}
\left(\pi^*\Omega\right)\big|_{U'} &= \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge \frac{d(y_{k+1}x_n)}{y_{k+1}x_n} \wedge \cdots \wedge \frac{d(y_{n-1}x_n)}{y_{n-1}x_n} \wedge \frac{dx_n}{x_n} \\
&= \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge \frac{dy_{k+1}}{y_{k+1}} \wedge \cdots \wedge \frac{dy_{n-1}}{y_{n-1}} \wedge \frac{dx_n}{x_n}.
\end{aligned}
$$

In particular, the pullback has a simple pole along $\{x_n = 0\} = E \cap U'$, as desired. We compute the residue along $E$:

$$
\begin{aligned}
\left(\operatorname{Res}_E(\pi^*\Omega)\right)\big|_{E \cap U'} &= \operatorname{Res}_{\{x_n=0\}}\left((\pi^*\Omega)|_{U'}\right) \\
&= \left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k}\right) \wedge \left(\frac{dy_{k+1}}{y_{k+1}} \wedge \cdots \wedge \frac{dy_{n-1}}{y_{n-1}}\right) \\
&= \Omega(\mathbb{P}^k, \Delta^k)\big|_{X_0 \neq 0} \wedge \Omega(\mathbb{P}^{n-k-1}, \Delta^{n-k-1})\big|_{Y_n \neq 0} \\
&= \left(\Omega(\mathcal{P}_W) \wedge \Omega(\mathcal{P}^W)\right)\big|_{E \cap U'},
\end{aligned}
$$

where the third and fourth equalities are justified as follows. For $\mathbb{P}^k$ with homogeneous coordinates $X_0, \ldots, X_k$, the form $\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k}$ is the canonical form $\Omega(\mathbb{P}^k, \Delta^k)$ restricted to the chart $\mathbb{A}^k \cong \{X_0 \neq 0\} \subset \mathbb{P}^k$ with the standard orientation. Similarly, for $\mathbb{P}^{n-k-1}$ with homogeneous coordinates $Y_{k+1}, \ldots, Y_n$, the form $\frac{dy_{k+1}}{y_{k+1}} \wedge \cdots \wedge \frac{dy_{n-1}}{y_{n-1}}$ is the canonical form $\Omega(\mathbb{P}^{n-k-1}, \Delta^{n-k-1})$ restricted to the chart $\mathbb{A}^{n-k-1} \cong \{Y_n \neq 0\} \subset \mathbb{P}^k$ with the standard orientation. Under the natural identification $E \cong \mathbb{P}^k \times \mathbb{P}^{n-k-1}$, our open chart $E \cap U' \subset E$ corresponds precisely to the product of these charts $\mathbb{A}^k \times \mathbb{A}^{n-k-1} \subset \mathbb{P}^k \times \mathbb{P}^{n-k-1}$. Hence, the standard orientations on the two affine charts $\mathbb{A}^k$ and $\mathbb{A}^{n-k-1}$ induce the standard orientation on $\mathbb{A}^k \times \mathbb{A}^{n-k-1} = \mathbb{A}^{n-1} \cong E \cap U'$, which is the orientation induced by the restriction of our standard orientation on $U'$ to the boundary $E \cap U' = \{x_n = 0\}$ (see Remark 4.3.7). $\qquad\square$

*Proof of Proposition 4.3.9.* Fix a triangulation $\mathcal{T}$ of $\mathcal{P}$. By Proposition 4.3.6(b), we have

$$
\Omega(\mathbb{P}V, \mathcal{P}) = \sum_{T \in \mathcal{T}} \Omega(\mathbb{P}V, T).
$$

Let $\mathcal{T}^{\mathrm{full}} = \{T \in \mathcal{T} \mid \dim T_W = \dim \mathbb{P}W\}$. By Proposition 4.3.8, if $T \notin \mathcal{T}^{\mathrm{full}}$ then $\pi^*\Omega(\mathbb{P}V, T)$ has no pole along the exceptional divisor $E$. Thus, applying Lemma 4.3.12 to simplices in $\mathcal{T}^{\mathrm{full}}$, we find

$$
\operatorname{Res}_E(\pi^*\Omega(\mathbb{P}V, \mathcal{P})) = \sum_{T \in \mathcal{T}^{\mathrm{full}}} \operatorname{Res}_E\left(\pi^*\Omega(\mathbb{P}V, T)\right) = \sum_{T \in \mathcal{T}^{\mathrm{full}}} \Omega(\mathbb{P}W, T_W) \wedge \Omega(\mathbb{P}(V/W), T^W).
$$

Since $\dim \mathbb{P}W = \dim \mathcal{P}_W$, Lemma 4.3.5(a) implies that the collection $\mathcal{T}_W = \{T_W \mid T \in \mathcal{T}^{\mathrm{full}}\}$ is a triangulation of $\mathcal{P}_W$, and Lemma 4.3.5(b) implies that for any simplex $F$ in the

triangulation $\mathcal{T}_W$ of $\mathcal{P}_W$, the collection $\mathcal{T}^{W,F} = \{T^W \mid T \in \mathcal{T}^{\mathrm{full}} \text{ satisfying } T_W = F\}$ is a triangulation of $\mathcal{P}^W$. Thus, we compute that

$$
\begin{aligned}
\mathrm{Res}_E(\pi^*\Omega(\mathbb{P}V, \mathcal{P})) &= \sum_{T \in \mathcal{T}^{\mathrm{full}}} \Omega(\mathbb{P}W, T_W) \wedge \Omega(\mathbb{P}(V/W), T^W) \\
&= \sum_{F \in \mathcal{T}_W} \sum_{\substack{T \in \mathcal{T}^{\mathrm{full}} \\ T_W = F}} \Omega\left(\mathbb{P}W, T_W\right) \wedge \Omega\left(\mathbb{P}(V/W), T^W\right) \\
&= \sum_{F \in \mathcal{T}_W} \Omega(\mathbb{P}W, F) \wedge \left( \sum_{\substack{T \in \mathcal{T}^{\mathrm{full}} \\ T_W = F}} \Omega\left(\mathbb{P}(V/W), T^W\right) \right) \\
&= \sum_{F \in \mathcal{T}_W} \Omega(\mathbb{P}W, F) \wedge \Omega\left(\mathbb{P}(V/W), \mathcal{P}^W\right) \\
&= \left( \sum_{F \in \mathcal{T}_W} \Omega(\mathbb{P}W, F) \right) \wedge \Omega\left(\mathbb{P}(V/W), \mathcal{P}^W\right) \\
&= \Omega\left(\mathbb{P}W, \mathcal{P}_W\right) \wedge \Omega\left(\mathbb{P}(V/W), \mathcal{P}^W\right).
\end{aligned}
$$

Here, for the fourth and sixth equalities, we used Proposition 4.3.6(b) applied to the triangulations $\mathcal{T}_W$ and $\mathcal{T}^{W,F}$ of $(\mathbb{P}W, \mathcal{P}_W)$ and $(\mathbb{P}(V/W), \mathcal{P}^W)$, respectively, which are positive geometries because $\mathcal{P}_W \subset \mathbb{P}W$ and $\mathcal{P}^W \subset \mathbb{P}(V/W)$ are full-dimensional polytopes.

Lastly, that $(E, E_{\geq 0})$ is a positive geometry with the canonical form $\mathrm{Res}_E(\pi^*\Omega(\mathbb{P}V, \mathcal{P}))$ now follows from Lemma 4.3.13 below, which states that products of positive geometries are positive geometries. □

**Lemma 4.3.13.** *[ABL17, Section 2.3] If $(X, X_{\geq 0})$ and $(Y, Y_{\geq 0})$ are positive geometries with canonical forms $\Omega(X_{\geq 0})$ and $\Omega(Y_{\geq 0})$, respectively, then $(X \times Y, X_{\geq 0} \times Y_{\geq 0})$ is a positive geometry with canonical form $\Omega(X_{\geq 0}) \wedge \Omega(Y_{\geq 0})$.*

We will use Lemma 4.3.13 again in Theorem 4.4.5.

## 4.4 Blowing up a sequence of faces

In this section we prove Theorem 4.1.4, our main result. In Section 4.4.1, we prepare by recording facts about the boundary structure of a wonderful compactification $X^{\mathcal{B}}$ and a wondertope $\widetilde{\mathcal{P}}^{\mathcal{B}}$. In Section 4.4.2, we prove the main theorem via Theorem 4.3.1 and induction.

### 4.4.1 Wonderful compactifications and their boundaries

Let $\mathcal{B}$ be a set of proper linear subvarieties of $\mathbb{P}V$. We fix an ordering $\mathcal{B} = \{\mathbb{P}W_1, \ldots, \mathbb{P}W_k\}$ such that $i \leq j$ if $W_i \subseteq W_j$, and consider the sequential blow-up

$$
\pi_{\mathcal{B}} : \mathbb{P}V^{\mathcal{B}} = \mathrm{Bl}_{\mathbb{P}W_k}(\cdots (\mathrm{Bl}_{\mathbb{P}W_2}(\mathrm{Bl}_{\mathbb{P}W_1} \mathbb{P}V)) \cdots) \to \mathbb{P}V.
$$

(As before, we repurpose the notation to write $\mathbb{P}W_2$ also for its strict transform in $\mathrm{Bl}_{\mathbb{P}W_1} \mathbb{P}V$, and similarly for $\mathbb{P}W_3, \ldots, \mathbb{P}W_k$).

Assume now that $\mathcal{B}$ is a building set (as in Section 1.2). In this case, De Concini and Procesi called $X^{\mathcal{B}}$ the *wonderful compactification* of the arrangement complement $\mathbb{P}V \setminus (\bigcup \mathcal{B})$, and showed that its boundary has the following structure. For $\mathbb{P}W \in \mathcal{B}$, define

$$\mathcal{B}_W = \{F \cap \mathbb{P}W \mid F \in \mathcal{B} \text{ and } F \not\supseteq \mathbb{P}W\} \quad \text{and}$$
$$\mathcal{B}^W = \{\mathbb{P}(W'/W) \subseteq \mathbb{P}(V/W) \mid \mathbb{P}W' \in \mathcal{B} \text{ and } W' \supseteq W\},$$

and let $E_W \subset X^{\mathcal{B}}$ be (the strict transform of) the exceptional divisor of the blow-up at $\mathbb{P}W$.

**Proposition 4.4.1.** *[DP95, Theorem 4.3] For $\mathbb{P}W \in \mathcal{B}$, both $\mathcal{B}_W$ and $\mathcal{B}^W$ are building sets in $\mathbb{P}W$ and $\mathbb{P}(V/W)$, respectively, and there is a natural isomorphism*

$$E_W \cong \mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W}$$

*along with a commutative diagram of natural maps*

$$
\begin{array}{ccc}
X^{\mathcal{B}} & \xleftarrow{\quad \supset \quad} & E_W \cong \mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W} \\
{\scriptstyle \pi_{\mathcal{B}}}\downarrow & & \downarrow {\scriptstyle \pi_{\mathcal{B}_W} \times \pi_{\mathcal{B}^W}} \\
\mathbb{P}V \xleftarrow[\pi_W]{} \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V & \xleftarrow{\quad \supset \quad} & E \cong \mathbb{P}W \times \mathbb{P}(V/W)
\end{array}
\quad .
$$

In analogy with Proposition 4.4.1, we show that a wondertope has a similar boundary structure, as follows. Let $\mathcal{P} \subset \mathbb{P}V$ be a full-dimensional polytope, and assume further now that $\mathcal{B}$ satisfies the property:

for every $F \in \mathcal{B}$, the intersection $F \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$.

Let $\widetilde{\mathcal{P}}^{\mathcal{B}}$ be the Euclidean closure of $\pi_{\mathcal{B}}^{-1}(\mathrm{int}(\mathcal{P}))$ in $X^{\mathcal{B}}(\mathbb{R})$. This is technically not a wondertope since we have not yet assumed that all facet hyperplanes of $\mathcal{P}$ are in $\mathcal{B}$. In the following two lemmas, we will record some properties of the boundary structure of $\widetilde{\mathcal{P}}^{\mathcal{B}}$.

For $\mathbb{P}H \subset \mathbb{P}V$ a hyperplane such that $\mathcal{P}_H = \mathcal{P} \cap \mathbb{P}H$ is a facet of $\mathcal{P}$, we define

$$\mathcal{B}_H = \{F \cap \mathbb{P}H \mid F \in \mathcal{B} \text{ and } F \not\supseteq \mathbb{P}H\},$$

whose ordering is inherited from the order on $\mathcal{B}$. Let $\widetilde{\mathbb{P}H}$ be the strict transform of $\mathbb{P}H$ under $\pi_{\mathcal{B}}$, which by the universal property of blow-ups is isomorphic to the sequential blow-up

$$\pi_{\mathcal{B}_H} : \mathbb{P}H^{\mathcal{B}_H} \to \mathbb{P}H.$$

We define

$$\widetilde{\mathbb{P}H}_{\geq 0} = \text{the Euclidean closure of the interior of } \widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H} \text{ inside } \widetilde{\mathbb{P}H}(\mathbb{R}), \quad \text{and}$$

$$\widetilde{\mathcal{P}_H}^{\mathcal{B}_H} = \text{the Euclidean closure of } \pi_{\mathcal{B}_H}^{-1}(\mathrm{int}(\mathcal{P}_H)) \text{ in } \mathbb{P}H^{\mathcal{B}_H}(\mathbb{R}).$$

We caution that $\mathcal{B}_H$ may not be a building set in $\mathbb{P}H$, so we will soon impose an additional condition on $\mathcal{B}$ (see Definition 4.4.4) when we prove Theorem 4.1.4. Moreover, we caution that taking Euclidean closure of the interior in the definition of $\widetilde{\mathbb{P}H}_{\geq 0}$ is necessary for the following lemma: otherwise, the intersection $\widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H}$ (without taking the closure of the interior) may not be equal to $\widetilde{\mathcal{P}_H}^{\mathcal{B}_H}$, so the following lemma would fail. This is illustrated in Example 4.5.5.

**Lemma 4.4.2.** *Under the isomorphism* $\widetilde{\mathbb{P}H} \cong \mathbb{P}H^{\mathcal{B}_H}$, *we have* $\widetilde{\mathbb{P}H}_{\geq 0} = \widetilde{\mathcal{P}_H}^{\mathcal{B}_H}$.

*Proof.* Since the exceptional locus intersects $\widetilde{\mathbb{P}H}$ in a subset of codimension at least 1, we can detect (a dense open subset of) the interior of $\widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H}$ inside $\widetilde{\mathbb{P}H}(\mathbb{R})$ by looking away from the exceptional locus. We note that $\pi_{\mathcal{B}}$ is an isomorphism away from the exceptional locus. Therefore, outside of the exceptional locus, we know that $\widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H}$ is given by $\pi_{\mathcal{B}}^{-1}(\mathcal{P} \cap H) = \pi_{\mathcal{B}}^{-1}(\mathcal{P}_H)$. So a dense open subset of $\operatorname{int}(\widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H})$ is given by $\pi_{\mathcal{B}}^{-1}(\operatorname{int}(\mathcal{P}_H))$. We know that, restricted to $\widetilde{\mathbb{P}H}$, our wonderful compactification $\pi_{\mathcal{B}}$ looks like $\pi_{\mathcal{B}_H} : \widetilde{\mathbb{P}H} \cong \mathbb{P}H^{\mathcal{B}_H} \to \mathbb{P}H$, so $\pi_{\mathcal{B}}^{-1}(\operatorname{int}(\mathcal{P}_H)) = \pi_{\mathcal{B}_H}^{-1}(\operatorname{int}(\mathcal{P}_H)) = \operatorname{int}(\widetilde{\mathcal{P}_H}^{\mathcal{B}_H})$. Therefore,

$$\widetilde{\mathbb{P}H}_{\geq 0} = \overline{\operatorname{int}(\widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H})}^{\text{closure in } \widetilde{\mathbb{P}H}} = \overline{\operatorname{int}(\widetilde{\mathcal{P}_H}^{\mathcal{B}_H})}^{\text{closure in } \mathbb{P}H^{\mathcal{B}_H}} = \widetilde{\mathcal{P}_H}^{\mathcal{B}_H}. \qquad \square$$

Let $\mathbb{P}W \in \mathcal{B}$ be a linear subspace intersecting our polytope $\mathcal{P}$ along a face $\mathcal{P}_W = \mathcal{P} \cap \mathbb{P}W$, and let $E_W$ be (the strict transform of) the exceptional divisor in $X^{\mathcal{B}}$ corresponding to $\mathbb{P}W$. Let

$$E_{W,\geq 0} = \text{the Euclidean closure of the interior of } \widetilde{\mathcal{P}}^{\mathcal{B}} \cap E_W \text{ inside } E_W(\mathbb{R}).$$

**Lemma 4.4.3.** *Under the isomorphism* $E_W \cong (\mathbb{P}W)^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W}$, *we have*

$$E_{W,\geq 0} = \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \times \widetilde{\mathcal{P}W}^{\mathcal{B}^W}.$$

*In particular, the subset* $E_{W,\geq 0}$ *is full-dimensional in* $E_W$ *if and only if* $\dim \mathcal{P}_W = \dim \mathbb{P}W$.

*Proof.* As in Lemma 4.4.2, because the rest of the exceptional locus intersects $E_W$ in a subset of codimension at least 1, we can detect a dense open subset of $\operatorname{int}(\widetilde{\mathcal{P}}^{\mathcal{B}} \cap E_W)$ by looking away from this additional exceptional locus. Away from the exceptional locus corresponding to $\mathcal{B} \setminus \{\mathbb{P}W\}$, our blow-up $\pi_{\mathcal{B}}$ looks like (a dense open subset of) a single blow-up

$$\pi_W : \operatorname{Bl}_{\mathbb{P}W} \mathbb{P}V \to \mathbb{P}V,$$

with exceptional divisor $E \cong \mathbb{P}W \times \mathbb{P}(V/W)$. Therefore, we know that, away from the exceptional locus of $\mathcal{B} \setminus \{\mathbb{P}W\}$, $\widetilde{\mathcal{P}}^{\mathcal{B}} \cap E_W$ is given by $\widetilde{\mathcal{P}}^{\{\mathbb{P}W\}} \cap E$. So (a dense open subset of) the interior $\operatorname{int}(\widetilde{\mathcal{P}}^{\mathcal{B}} \cap E_W)$ is given by (a dense open subset of) $\operatorname{int}(\mathcal{P}_W) \times \operatorname{int}(\mathcal{P}^W) \subset$

$\mathbb{P}W \times \mathbb{P}(V/W) \cong E$, according to Proposition 4.3.3. In particular, this is empty if $\mathcal{P}_W$ is not full-dimensional in $\mathbb{P}W$. So let's assume that $\dim \mathcal{P}_W = \dim \mathbb{P}W$. Under the canonical map

$$\pi_{B_W} \times \pi_{\mathcal{B}^W} : \left( E_W \cong \mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W} \right) \to \left( E \cong \mathbb{P}W \times \mathbb{P}(V/W) \right),$$

the inverse image of $\operatorname{int}(\mathcal{P}_W) \times \operatorname{int}(\mathcal{P}^W)$ is precisely $\operatorname{int}\left( \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \right) \times \operatorname{int}\left( \widetilde{\mathcal{P}^W}^{\mathcal{B}^W} \right)$, and so

$$E_{W,\geq 0} = \overline{\operatorname{int}\left( \widetilde{\mathcal{P}}^W \cap E_W \right)}^{\text{closure in } E_W}$$

$$= \overline{\operatorname{int}\left( \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \right) \times \operatorname{int}\left( \widetilde{\mathcal{P}^W}^{\mathcal{B}^W} \right)}^{\text{closure in } \mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W}}$$

$$= \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \times \widetilde{\mathcal{P}^W}^{\mathcal{B}^W}. \quad \square$$

### 4.4.2  Proof of the main theorem

We are now ready to prove Theorem 4.1.4. We in fact prove a stronger version, stated as follows. Let $\mathcal{P} \subset \mathbb{P}V$ be a full-dimensional polytope. Let $\mathcal{B}$ be a set of proper linear subvarieties of $\mathbb{P}V$ satisfying the following properties stated in the introduction:

- for every $F \in \mathcal{B}$, the intersection $F \cap \mathcal{P}$ is a (possibly empty) face of $\mathcal{P}$, and

- $\mathcal{B}$ is a building set.

Furthermore, we relax the condition (2) stated in the introduction to the following.

**Definition 4.4.4.** We say that such a set $\mathcal{B}$ is *well-adapted to* $\mathcal{P}$ if either $\dim \mathbb{P}V = 1$ or, otherwise, $\mathcal{B}$ further satisfies the recursive property:

- for every facet hyperplane, i.e. a hyperplane $\mathbb{P}H \subset \mathbb{P}V$ such that $\mathcal{P}_H = \mathcal{P} \cap \mathbb{P}H$ is a facet of $\mathcal{P}$, the set $\mathcal{B}_H$ is a building set in $\mathbb{P}H$ and is well-adapted to $\mathcal{P}_H$.

We prove the following theorem.

**Theorem 4.4.5.** *Let $\mathcal{B}$ be well-adapted to $\mathcal{P}$. Then, the pair $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ is a positive geometry whose canonical form is the pullback $\pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P})$ of the canonical form of $(\mathbb{P}V, \mathcal{P})$. The boundary components of $(X^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ are the exceptional divisors $E_F$ for $F \in \mathcal{B}$ such that $\dim(\mathcal{P} \cap F) = \dim F$, together with the strict transforms of the facet hyperplanes of $\mathcal{P}$ that are not in $\mathcal{B}$.*

*Proof of Theorem 4.1.4 from Theorem 4.4.5.* If $\mathcal{B}$ contains all the facet hyperplanes of $\mathcal{P}$, then $\mathcal{B}_H$ is a building set in $\mathbb{P}H$ by Proposition 4.4.1. Moreover, $\mathcal{B}_H$ contains all the facet hyperplanes of $\mathcal{P}_H$, since a facet hyperplane of $\mathcal{P}_H$ is an intersection of $\mathbb{P}H$ with another facet hyperplane of $\mathcal{P}$. Hence, the set $\mathcal{B}$ is well-adapted to $\mathcal{P}$ in this case. In particular, Theorem 4.4.5 contains Theorem 4.1.4 as a special case. $\quad \square$

**Remark 4.4.6.** If $\mathcal{B}$ consists of linear subspaces that intersect $\mathcal{P}$ along (possibly empty) faces and that are closed under intersection (i.e. if $F, F' \in \mathcal{B}$ then $F \cap F' \in \mathcal{B}$), then $\mathcal{B}$ is a building set, and $\mathcal{B}_H$ is intersection-closed for any hyperplane $\mathbb{P}H$. Hence, the set $\mathcal{B}$ is well-adapted to $\mathcal{P}$. In this intersection-closed setting, Brown [Bro25, Section 5] calls the wondertope $\widetilde{\mathcal{P}}^{\mathcal{B}}$ a "blow-up of a polyhedral linear configuration" and describes the structure of its boundary divisors.

Theorem 4.4.5 fails without the hypothesis that $\mathcal{B}$ is well-adapted to $\mathcal{P}$, even if $\mathcal{B}$ is a building set; see Example 4.5.7.

*Proof of Theorem 4.4.5.* We induct on $\dim \mathbb{P}V$. The base case $\dim \mathbb{P}V = 1$ holds trivially since the blow-down map is an isomorphism onto all of $\mathbb{P}V$ in that case.

For the general case, we must verify that the residues of $\pi^* \Omega(\mathbb{P}V, \mathcal{P})$ are the canonical forms of the boundary components of $\partial \widetilde{\mathcal{P}}$. The boundary components come in two cases: (i) the strict transform $\widetilde{\mathbb{P}H}$ of a hyperplane $\mathbb{P}H \subset \mathbb{P}V$ such that $\mathcal{P}_H$ is a facet of $\mathcal{P}$ and $\mathbb{P}H \notin \mathcal{B}$, and (ii) the exceptional divisor $E_W$ for $\mathbb{P}W \in \mathcal{B}$. That there are no more boundary components follows from the fact that the boundary components of $\partial \mathcal{P}$ are its facet hyperplanes, $\pi_{\mathcal{B}}$ is an isomorphism outside of the exceptional locus $\bigcup_{\mathbb{P}W \in \mathcal{B}} E_W$, and $\pi_{\mathcal{B}}(\widetilde{\mathcal{P}}^{\mathcal{B}}) = \mathcal{P}$.

In the case of (i), by the universal property of blow-ups, we have $\widetilde{\mathbb{P}H} \cong \mathbb{P}H^{\mathcal{B}_H}$, with $\pi_{\mathcal{B}H}$ identified with the restriction $\pi_{\mathcal{B}}|_{\widetilde{\mathbb{P}H}}$. By the induction hypothesis, the pair $(\widetilde{\mathbb{P}H}, \widetilde{\mathcal{P}_H}^{\mathcal{B}_H})$ is a positive geometry with canonical form $\pi_{\mathcal{B}_H}^* \Omega(\mathbb{P}H, \mathcal{P}_H)$. The pair is also a boundary component of $(\mathbb{P}V^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ by Lemma 4.4.2. To verify that $\mathrm{Res}_{\widetilde{\mathbb{P}H}} \pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P}) = \pi_{\mathcal{B}_H}^* \Omega(\mathbb{P}H, \mathcal{P}_H)$, by Lemma 4.3.10, it suffices to do so on the Zariski dense open subset $U = \mathbb{P}V^{\mathcal{B}} \setminus (\bigcup_{\mathbb{P}W \in \mathcal{B}} E_W)$, which restricts to a Zariski dense open subset in $\widetilde{\mathbb{P}H}$. Indeed, since $\pi_{\mathcal{B}}$ is an isomorphism on $U$, we find that over $U$, we have

$$\mathrm{Res}_{\widetilde{\mathbb{P}H}}(\pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P})) = \pi_{\mathcal{B}_H}^* \big( \mathrm{Res}_{\mathbb{P}H} \Omega(\mathbb{P}V, \mathcal{P}) \big) = \pi_{\mathcal{B}_H}^* \big( \Omega(\mathbb{P}H, \mathcal{P}_H) \big).$$

In the case of (ii), we first note that it suffices to consider only $\mathbb{P}W \in \mathcal{B}$ such that $\dim \mathcal{P}_W = \dim \mathbb{P}W$. Indeed, if $\dim \mathcal{P}_W < \dim \mathbb{P}W$ by Proposition 4.3.8 and the commutative diagram in Proposition 4.4.1, we find that $\pi^* \Omega(\mathbb{P}V, \mathcal{P})$ has no pole along $E_W$. Meanwhile, Lemma 4.4.3 implies that $E_{W, \geq 0}$ is not full-dimensional if $\dim \mathcal{P}_W < \dim \mathbb{P}W$.

When $\dim \mathcal{P}_W = \dim \mathbb{P}W$, by the induction hypothesis and Lemma 4.3.13, the pair

$$\left( \mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W}, \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \times \widetilde{\mathcal{P}^W}^{\mathcal{B}^W} \right)$$

is a positive geometry. By Proposition 4.4.1 and Lemma 4.4.3, we have an isomorphism

$$(E_W, E_{W, \geq 0}) \cong \left( \mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W}, \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \times \widetilde{\mathcal{P}^W}^{\mathcal{B}^W} \right),$$

and in particular $(E, E_{W, \geq 0})$ is a boundary component of $(\mathbb{P}V^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$.

To verify that $\Omega(E_W, E_{W,\geq 0}) = \mathrm{Res}_{E_W} \pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P})$, by Lemma 4.3.10 it suffices to do so on the Zariski dense open subset $U_W = \mathbb{P}V^{\mathcal{B}} \setminus (\bigcup_{\substack{\mathbb{P}W' \in \mathcal{B} \\ \mathbb{P}W' \neq \mathbb{P}W}} E_{W'})$, which restricts to a Zariski dense open subset in $E_W$. Indeed, on $U_W$, Proposition 4.4.1 identifies $E_W$ with the exceptional divisor $E$ of the blow-up $\pi_W : \mathrm{Bl}_{\mathbb{P}W} \mathbb{P}V \to \mathbb{P}V$, and hence, over $U_W$, we have

$$
\begin{aligned}
\mathrm{Res}_{E_W} \pi_{\mathcal{B}}^* \Omega(\mathbb{P}V, \mathcal{P}) &= \mathrm{Res}_E \pi_W^* \Omega(\mathbb{P}V, \mathcal{P}) \\
&= \Omega\left(\mathbb{P}W \times \mathbb{P}(V/W), \mathcal{P}_W \times \mathcal{P}^W\right) \\
&= \Omega\left(\mathbb{P}W^{\mathcal{B}_W} \times \mathbb{P}(V/W)^{\mathcal{B}^W}, \widetilde{\mathcal{P}_W}^{\mathcal{B}_W} \times \widetilde{\mathcal{P}^W}^{\mathcal{B}^W}\right)
\end{aligned}
$$

where the second equality follows from Theorem 4.3.1, and the last equality follows from that the pair $(E, E_{\geq 0})$ in Theorem 4.3.1 coincides with the pair $(E_W, E_{W,\geq 0})$ on $U_W$.

Lastly, the uniqueness of the canonical form follows from the rationality of $X^{\mathcal{B}}$, similarly to the argument at the end of the proof of Theorem 4.3.1. □

## 4.5 Examples and questions

In Section 4.5.1, we give a detailed account of how our Theorem 4.1.4 recovers a result about the moduli space of pointed stable rational curves. In Section 4.5.2, we collect several examples where a blow-up of a positive geometry does not result in a positive geometry. In Section 4.5.3, we detail an example that both illustrates Theorem 4.3.1 and provides ways to produce pathologies. We finish with an example with a view towards a generalization of Theorem 4.4.5 to a setting beyond polytopes.

### 4.5.1 Motivating example: $\overline{M}_{0,n+1}$ and the braid arrangement

Let $\overline{M}_{0,n+1}$ be the moduli space of stable $(n+1)$-pointed rational curves; we point to [KV07, Chapter 1] as a reference. In [AHL21a, Proposition 8.2], the authors show that $(\overline{M}_{0,n+1}, (\overline{M}_{0,n+1})_{\geq 0})$ is a positive geometry, using theory developed in [BCS10, Proposition 2.7]. Our work provides a self-contained proof of this fact by viewing $\overline{M}_{0,n+1}$ as a certain wonderful compactification of the braid arrangement complement.

#### $M_{0,n+1}$ as the braid arrangement complement

The space $M_{0,n+1}$ is the moduli space of $n+1$ distinct points $z_0, z_1, \ldots, z_n$ in $\mathbb{P}^1$, up to projective linear transformations of $\mathbb{P}^1$. Fixing $(z_0, z_{n-1}, z_n) = (0, 1, \infty)$ allows us to write

$$
\begin{aligned}
M_{0,n+1} &= \left\{ (z_1, z_2, \ldots, z_{n-2}) \in (\mathbb{P}^1)^{n-2} : z_i \neq 0, 1, \infty \text{ and } z_i \neq z_j \text{ for } i \neq j \right\} \\
&\cong \left\{ (z_1, z_2, \ldots, z_{n-2}) \in \mathbb{A}^{n-2} : z_i \neq 0, 1 \text{ and } z_i \neq z_j \text{ for } i \neq j \right\}.
\end{aligned}
$$

Let $V = \mathbb{R}^n / \mathbb{R} \cdot (1, 1, \ldots, 1)$. Recall that the *rank $(n-1)$ braid arrangement* $A_{n-1}$ consists of the image of the hyperplanes $\{x_i - x_j = 0\} \subset \mathbb{R}^n$ in $\mathbb{P}V$ for $1 \leq i < j \leq n$ (see for example [Sta07, Ch. 1].)

It is well-known that $M_{0,n+1}$ is isomorphic to the complement $C(A_{n-1}) := \mathbb{P}V \setminus A_{n-1}$ of the braid arrangement. We will go through this isomorphism in detail because it is useful in deriving the *Parke-Taylor form* from Corollary 4.1.5.

Consider $V \cong \mathbb{R}^{n-1}$ with basis given by the images $\overline{e_i} \in V$ of the standard basis vectors $e_i \in \mathbb{R}^n$ for $1 \leq i < n$. The projectivization $\mathbb{P}V \cong \mathbb{P}^{n-2}$ thus consists of non-zero points $y_1\overline{e_1} + \cdots + y_{n-1}\overline{e_{n-1}}$, modulo scalar multiplication. The image of the hyperplanes $\{x_i - x_j = 0\} \subset \mathbb{R}^n$ in $\mathbb{P}V$ are the hyperplanes $\{y_i = 0\}$ for $1 \leq i \leq n-1$ and $\{y_i - y_j = 0\}$ for $1 \leq i < j \leq n-1$. Therefore, the complement $C(A_{n-1})$ can be written as

$$C(A_{n-1}) = \left\{ [y_1\overline{e_1} + \cdots + y_{n-1}\overline{e_{n-1}}] \in \mathbb{P}V : y_i \neq 0 \text{ and } y_i \neq y_j \text{ for } i \neq j \right\}$$
$$\cong \left\{ [y_1 : \cdots : y_{n-1}] \in \mathbb{P}^{n-2} : y_i \neq 0 \text{ and } y_i \neq y_j \text{ for } i \neq j \right\}.$$

The isomorphism $C(A_{n-1}) \cong M_{0,n+1}$ then follows via the change of coordinates $z_i = y_i/y_{n-1}$ for $1 \leq i \leq n-2$. Indeed, we obtain:

$$C(A_{n-1}) \cong \left\{ [z_1 : \cdots : z_{n-2} : 1] \in \mathbb{P}^{n-2} : z_i \neq 0, 1 \text{ and } z_i \neq z_j \text{ for } i \neq j \right\}$$
$$\cong \left\{ (z_1, z_2, \ldots, z_{n-2}) \in \mathbb{R}^{n-2} : z_i \neq 0, 1 \text{ and } z_i \neq z_j \text{ for } i \neq j \right\}.$$

**Wonderful Compactification**

Under the correct choice of building set for $C(A_{n-1})$, the corresponding wonderful compactification is isomorphic to the Deligne–Knudsen–Mumford compactification $\overline{M}_{0,n+1}$ [DP95, Remark 4.3.(3)]. This building set is called the *minimal building set $\mathcal{B}^{\min}$*, defined as

$$\mathcal{B}^{\min} := \left\{ \mathbb{P}W : W = \{x_{j_1} = x_{j_2} = \cdots = x_{j_k}\} \text{ for } 0 \leq j_1 < \cdots < j_k \leq n \text{ and } k \geq 2 \right\}.$$

There is a nice combinatorial restatement of this story. Let $\mathscr{L}(A_{n-1})$ be the poset with elements given by intersections of hyperplanes in $A_{n-1}$, ordered by reverse inclusion. It is in fact a lattice, and is well known to be isomorphic to $\Pi_n$, the lattice of set partitions of $[n] := \{1, 2, \cdots, n\}$, ordered by reverse refinement. Under this isomorphism, the set partition $\pi$ corresponds to the subspace $W$ where $x_i = x_j$ in $W$ if and only if $i, j$ are in the same block in $\pi$. For example, the set partition $\{13 \mid 246 \mid 5\}$ corresponds to the subspace in $\mathscr{L}(A_5)$ where $x_1 = x_3$ and $x_2 = x_4 = x_6$.

We will repurpose notation and refer interchangeably to $W \in \mathscr{L}(A_{n-1})$ and the corresponding set partition in $\Pi_n$. In this language, the minimal building set is

$$\mathcal{B}^{\min} = \left\{ \{j_1, j_2, \cdots, j_k \mid i_1 \mid \cdots \mid i_\ell\} : k \geq 2 \right\}.$$

Figure 4.2 shows $\mathscr{L}(A_3) \cong \Pi_4$, with the elements of $\mathcal{B}^{\min}$ colored in red.
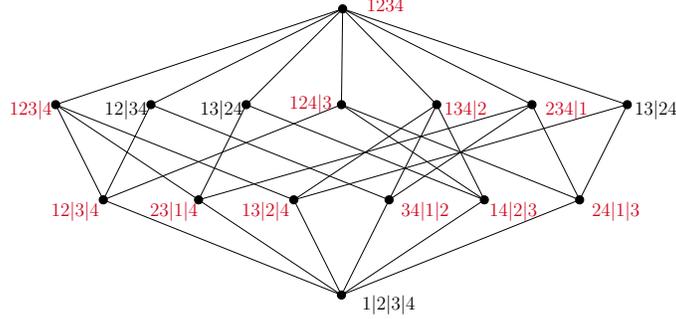
Figure 4.2: The lattice of flats $\mathscr{L}(A_3) \cong \Pi_4$. The flats in $\mathcal{B}^{\min}$ are drawn in red.

The boundary components of $\overline{M}_{0,n+1}$ are well-studied. In particular, there are $2^n - n - 2$ divisors, each of which factors as a product $\overline{M}_{0,k} \times \overline{M}_{0,n+3-k}$ for some $3 \leq k \leq n$. In the language of Section 4.4.1, this says that for any $\mathbb{P}W \in \mathcal{B}^{\min}$, both $\mathbb{P}W^{(\mathcal{B}^{\min})_W}$ and $\mathbb{P}(V/W)^{(\mathcal{B}^{\min})^W}$ are isomorphic to copies of $\overline{M}_{0,k}$ for some $3 \leq k \leq n$. From the perspective of hyperplane arrangements, this is equivalent to the statement that the restriction and contraction arrangements of $A_{n-1}$ by $W \in \mathcal{B}^{\min}$ are isomorphic to smaller braid arrangements.

## Wondertopes and the Parke-Taylor form

There are $n!/2$ regions in $C(A_{n-1}) \cong M_{0,n+1}$. The symmetric group acts transitively on these regions by permuting the standard basis vectors of $\mathbb{R}^n$, which gives an automorphism of $\mathbb{P}V$ preserving the hyperplanes in $A_{n-1}$. Thus, without loss of generality, we need only consider a single region in either space.

Each region in $M_{0,n+1}$ corresponds to an ordering of the points $z_0, z_1, \ldots, z_n$ on $\mathbb{P}^1(\mathbb{R})$, up to orientation. As before, fix $(z_0, z_{n-1}, z_n) = (0, 1, \infty)$ and consider the region

$$(M_{0,n+1})_{>0} = \{0 < z_1 < z_2 < \cdots < z_{n-2} < 1\}. \tag{4.2}$$

To see this region as a simplex, define new coordinates $x_i$ for $1 \leq i \leq n-2$ by:

$$x_i := \begin{cases} z_1 & \text{if } i = 1 \\ z_i - z_{i-1} & \text{if } 2 \leq i \leq n-2. \end{cases}$$

Under this linear change of coordinates, $M_{0,n+1}$ becomes

$$\begin{aligned} M_{0,n+1} &= \left\{(z_1, z_2, \ldots, z_{n-2}) \in \mathbb{R}^{n-2} : z_i \neq 0, 1 \text{ and } z_i \neq z_j \text{ for } i \neq j\right\} \\ &= \left\{(x_1, \ldots, x_{n-2}) \in \mathbb{R}^{n-2} : x_1 + \cdots + x_i \neq 1 \text{ and } x_i + x_{i+1} + \cdots + x_j \neq 0 \text{ for all } i \leq j\right\}. \end{aligned}$$

Then the region in (4.2) is the interior of a simplex in $\mathbb{R}^{n-2}$, defined via the following inequalities:

$$(M_{0,n+1})_{>0} = \left\{(x_1, x_2, \ldots, x_{n-2}) \in \mathbb{R}^{n-2} : x_i > 0 \text{ for all } i \text{ and } x_1 + x_2 + \cdots + x_{n-2} < 1\right\}.$$

Considering $\mathbb{R}^{n-2}$ as an affine chart of $\mathbb{P}^{n-2}(\mathbb{R})$, the Euclidean closure $(M_{0,n+1})_{\geq 0}$ of $(M_{0,n+1})_{>0}$ becomes a simplex in $\mathbb{P}^{n-2}(\mathbb{R})$. According to Proposition 4.3.6 (c), the pair $\big(\mathbb{P}^{n-2}, (M_{0,n+1})_{\geq 0}\big)$ is a positive geometry with canonical form given (up to sign) by

$$\Omega\big(\mathbb{P}^{n-2}, (M_{0,n+1})_{\geq 0}\big) = \frac{dx_1 dx_2 \cdots dx_{n-2}}{x_1 \cdots x_{n-2}(1 - x_1 - \cdots - x_{n-2})}. \tag{4.3}$$

Recall that $\overline{M}_{0,n+1}$ is the wonderful compactification of $M_{0,n+1}$ with respect to the minimal building set $\mathcal{B}^{\min}$ described in Section 4.5.1, and let $(\overline{M}_{0,n+1})_{\geq 0}$ be the strict transform of $(M_{0,n+1})_{\geq 0}$ under this compactification. Figure 4.3 shows an affine-local picture of $C(A_3) \cong M_{0,5}$ and $\overline{M}_{0,5}$, together with the regions $(M_{0,5})_{\geq 0}$ and $(\overline{M}_{0,5})_{\geq 0}$. (The singletons are dropped from the partition notation for simplicity.)
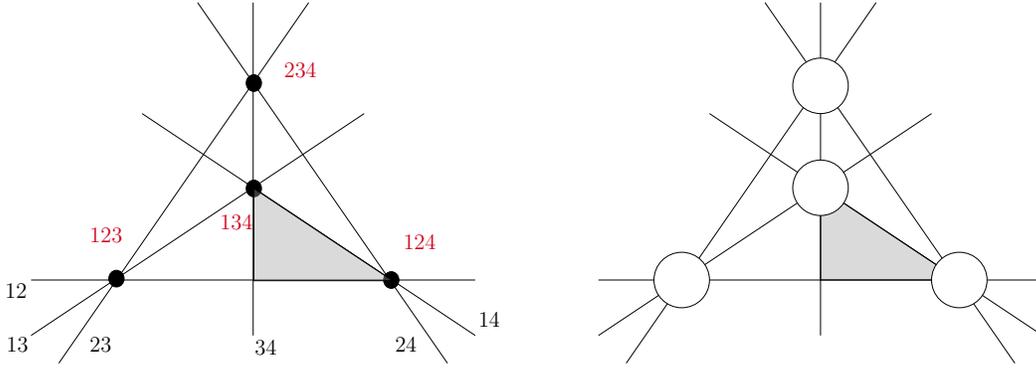


Figure 4.3: A region (a simplex) in $M_{0,5} \cong C(A_3)$ in $\mathbb{P}^2$ (left) and its wondertope $(\overline{M}_{0,5})_{\geq 0}$ (an associahedron) in $\overline{M}_{0,5}$ (right)

By Theorem 4.1.4, $\big(\overline{M}_{0,n+1}, (\overline{M}_{0,n+1})_{\geq 0}\big)$ is a positive geometry with canonical form

$$\Omega\big(\overline{M}_{0,n+1}, (\overline{M}_{0,n+1})_{\geq 0}\big) = \pi^* \, \Omega\big(\mathbb{P}^{n-2}, (M_{0,n+1})_{\geq 0}\big),$$

where $\pi$ denotes the blow-down map, as usual. Therefore, on an appropriate local chart where $\pi$ is the identity (i.e. away from the exceptional locus), the canonical form $\Omega\big(\overline{M}_{0,n+1}, (\overline{M}_{0,n+1})_{\geq 0}\big)$ is given by (4.3). Substituting back the original $z_i$ coordinates gives

$$\Omega\big(\overline{M}_{0,n+1}, (\overline{M}_{0,n+1})_{\geq 0}\big) = \frac{dz_1 \, dz_2 \cdots dz_{n-2}}{(z_1 - z_0)(z_2 - z_1)(z_3 - z_2) \cdots (z_{n-1} - z_{n-2})}, \tag{4.4}$$

where as before, $(z_0, z_{n-1}, z_n) = (0, 1, \infty)$. The form in (4.4) is known in the physics literature as the *Parke-Taylor form* [AHL21a, §8.2]. The Parke-Taylor form can also be written in terms of *dihedral coordinates* introduced by Brown in [Bro09]; see [Ear+25, §2] for a detailed example of $\overline{M}_{0,5}$.

It turns out that $(\overline{M}_{0,n+1})_{\geq 0}$ is homeomorphic, as a stratified space, to the face stratification of the *$n$-associahedron* [AHL21a], a convex $(n-2)$-dimensional polytope whose vertices

are in bijection with triangulations of an $(n+1)$-gon. The associahedron plays an important role in algebraic combinatorics, topology, and representation theory; an interested reader should look at [Pos09] and the references therein. For example, the wondertope $(\overline{M}_{0,5})_{\geq 0}$ in Figure 4.3 is combinatorially isomorphic to the 4-associahedron (a pentagon), whose vertices are in bijection with the triangulations of a 5-gon (also in this case a pentagon).

## 4.5.2 Non-examples

We collect a series of examples where a blow-up of a positive geometry is not a positive geometry, illustrating the necessity of the conditions in Theorem 4.4.5. To this end, we first note that $(\mathbb{P}^1, \mathbb{P}^1(\mathbb{R}))$ is not a positive geometry. Indeed, the nonnegative part $\mathbb{P}^1(\mathbb{R})$ has no boundary, so that the canonical form has no poles, but $\mathbb{P}^1$ has no nonzero 1-form with no poles. The pair $(\mathbb{P}^1, \mathbb{P}^1(\mathbb{R}))$ is instead a *pseudo-positive geometry* as in [ABL17, Section 2.2].

**Example 4.5.1.** We first give an example where we blow up a point in the boundary of a polytope which is not itself a face. Consider a triangle $T$ in $\mathbb{P}^2(\mathbb{R})$ and a point $p$ in the interior of an edge. Let $\pi : \mathrm{Bl}_p \, \mathbb{P}^2 \to \mathbb{P}^2$ be the blow-up.

Note that $T$ contains all normal directions (up to sign) at $p$. Thus, the Euclidean closure of $\pi^{-1}(\mathrm{int}(T))$, denoted $\widetilde{T}$, contains the entirety of the exceptional divisor $E(\mathbb{R})$. See Figure 4.4 for an illustration. In particular, we find that $E_{\geq 0}$, the Euclidean closure in $E(\mathbb{R})$ of the interior of $\widetilde{T} \cap E$, is all of $E(\mathbb{R}) \cong \mathbb{P}^1(\mathbb{R})$, so that the boundary component $(E, E_{\geq 0}) = (\mathbb{P}^1, \mathbb{P}^1(\mathbb{R}))$ is not a positive geometry. The pair $(\mathrm{Bl}_p \, \mathbb{P}^2, \widetilde{T})$ is thus not a positive geometry.
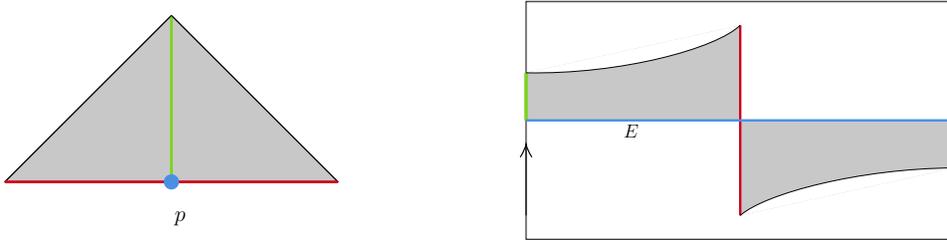


Figure 4.4: A triangle with point $p$ (left) and its blow-up (right)

$\diamond$

The next two examples illustrate that Theorem 4.4.5 does not generalize easily to the setting where the initial positive geometry is $(\mathbb{P}^n, \mathcal{P})$ for $\mathcal{P}$ a polyhedral complex instead of a polytope.

**Example 4.5.2.** Consider the positive geometry consisting of the three triangles in $\mathbb{P}^2(\mathbb{R})$ illustrated in Figure 4.5. Let us consider blowing up the point $p$. As in Example 4.5.1, every normal direction (up to sign) to $p$ is present in at least one polygon, so that once again we have $E_{\geq 0} = E(\mathbb{R}) \cong \mathbb{P}^1(\mathbb{R})$ and so $(E, E_{\geq 0})$ is not a positive geometry.

In this example, every log resolution of our positive geometry is *not* a positive geometry. Indeed, let $D$ be the union of the three lines through $p$, which is not a simple normal crossing divisor. By [Har77, Proposition V.5.2], any log resolution of the pair $(\mathbb{P}^2, D)$ must factor through the blow-up at $p$, which we have shown not to be a positive geometry. ◇
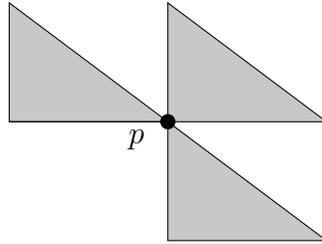


Figure 4.5: Three triangles meeting at a point

**Example 4.5.3.** Even if the polyhedral complex is connected in codimension 1, the resulting blow-up may not be a positive geometry. For example, consider four cubes arranged in a "staircase" and meeting at a point $p$, as in Figure 4.6. The boundary components of this polyhedral complex have a simple normal crossing at $p$, but the blow-up of $p$ is not a positive geometry. Indeed, because the polyhedral complex contains all normal directions to $p$, we again find $E_{\geq 0} = E(\mathbb{R})$. Moreover, in this case, $E(\mathbb{R}) \cong \mathbb{P}^2(\mathbb{R})$ is not orientable, so we do not even get a pseudo-positive geometry. ◇
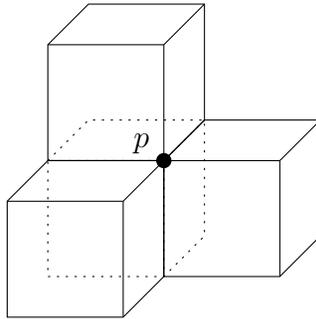


Figure 4.6: Four cubes meeting at a point

**Remark 4.5.4.** Examples 4.5.2 and 4.5.3 hint at the fact that, if one hopes to prove a more general statement about a positive geometry $(X, X_{\geq 0})$ remaining a positive geometry after blowing-up a boundary stratum, some "convexity condition" on the semialgebraic set $X_{\geq 0}$ is required. We caution that the notion of "corners" in [BD25, Definition 2.16] does not capture this necessary behavior in our setting as, for instance, the point $p$ in Example 4.5.3 is a corner.

### 4.5.3   A detailed example

We feature one example in detail, which will illustrate behaviors such as:

(i) Without the condition that a collection $\mathcal{B}$ includes all facet hyperplanes of $\mathcal{P}$, the boundary components of the pair $(\mathbb{P}V^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ may not be simple normal crossing, even though the pair is a positive geometry.

(ii) As cautioned above Lemma 4.4.2, for a facet hyperplane $\mathbb{P}H$ of $\mathcal{P}$, the intersection $\widetilde{\mathcal{P}}^{\mathcal{B}} \cap \widetilde{\mathbb{P}H}$ may not be equal to $\widetilde{\mathcal{P}_H}^{\mathcal{B}_H}$.

(iii) Without the condition that a collection $\mathcal{B}$ is well-adapted to a polytope $\mathcal{P}$, even if $\mathcal{B}$ is a building set, the pair $(\mathbb{P}V^{\mathcal{B}}, \widetilde{\mathcal{P}}^{\mathcal{B}})$ may not be a positive geometry.

All of our examples will be variations on a square pyramid $\mathcal{P}$ in $\mathbb{P}^3(\mathbb{R})$.

**Example 4.5.5.** This example will illustrate the behaviors (i) and (ii) mentioned above. Let $\mathcal{P}$ be a square pyramid in $\mathbb{P}^3(\mathbb{R})$. Let $p$ be the cone point of $\mathcal{P}$. Consider the blow-up $\mathrm{Bl}_Y \mathbb{P}^3$, where $Y$ is the line through points $p$ and $q$ as labelled in Figure 4.7. Note that the collection $\{Y\}$ is trivially a building set which is well-adapted to $\mathcal{P}$. However, this collection does not contain the facet hyperplanes of $\mathcal{P}$. In this case, the pair $(\mathrm{Bl}_Y \mathbb{P}^3, \widetilde{\mathcal{P}})$ is a positive geometry by Theorem 4.3.1, but we claim that its boundary components do not form a simple normal crossing divisor, and that the face poset of $\widetilde{\mathcal{P}}$ is not isomorphic to the face poset of any polytope.

Let $F_1, \ldots, F_4$ be the four triangles of the pyramid $\mathcal{P}$, with $F_1, F_2$ in the back and $F_3, F_4$ in the front. The exceptional divisor $E$ of the blow-up is isomorphic to $Y \times \mathbb{P}^1$. Using Proposition 4.3.3 to compute the strict transforms $\widetilde{\mathcal{P}}, \widetilde{F_1}, \ldots, \widetilde{F_4}$ of the polytopes, we find that they are as pictured in the right side of Figure 4.7. The nonnegative region $E_{\geq 0}$ is the quadrilateral in the back with the colored boundary.



Figure 4.7: A square pyramid with cone point $p$ (left) and its blow-up (right)

Let $H_i$ be the hyperplane containing $F_i$, and $\widetilde{H_i}$ its strict transform in the blow-up. Under the isomorphism $E \cong Y \times \mathbb{P}^1$, we have $\widetilde{H_3} \cap E = \widetilde{H_4} \cap E = \{p\} \times \mathbb{P}^1$, since $H_3$ and $H_4$ are

hyperplanes both intersecting $Y$ only at $p$. When restricted to $\widetilde{\mathcal{P}}$, these intersections are pictured in blue in Figure 4.7, which is a union of boundaries of $\widetilde{F_3}$ and $\widetilde{F_4}$. In particular, the positive geometry of $\widetilde{\mathcal{P}}$ has three 2-dimensional boundary components $\widetilde{H_3}$, $\widetilde{H_4}$, and $E$ whose common intersection is the 1-dimensional $\{p\} \times \mathbb{P}^1$. Such an intersection is not simple normal crossing, and such behavior does not happen in a 3-dimensional polytope: no three facets of any 3-dimensional polytope meet in a common line.

Lastly, we see that $\widetilde{\mathcal{P}} \cap \widetilde{H_4}(\mathbb{R})$ is the region colored purple in Figure 4.8, which contains an additional segment in $E_{\geq 0}$ when compared to $\widetilde{\mathcal{P}_{H_4}}$, colored green, illustrating the behavior (ii). ◇
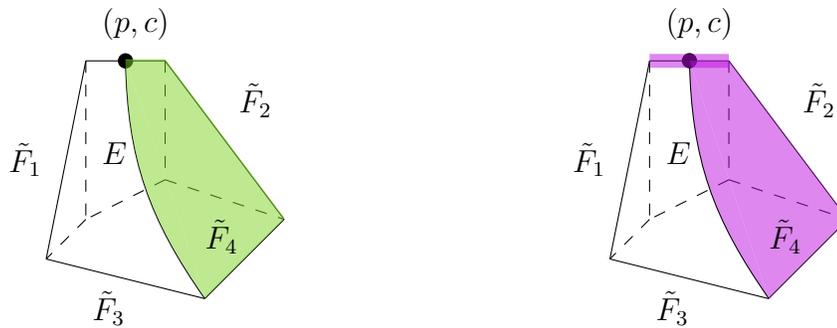


Figure 4.8: The semialgebraic subsets $\widetilde{\mathcal{P}_{H_4}}$ (left) and $\widetilde{\mathcal{P}} \cap \widetilde{H_4}(\mathbb{R})$ (right)

**Example 4.5.6.** Using the notation from Example 4.5.5, let us consider the collection $\{Y, Y'\}$ where $Y'$ is the line obtained by the intersection $H_3 \cap H_4$, i.e. the "opposite line" to the line $Y$ in the pyramid. Note that this collection is not a building set, since the intersection $Y \cap Y' = \{p\}$ has codimension 3 in $\mathbb{P}^3$, whereas $Y$ and $Y'$ each have codimension 2. In the second blow-up of the sequential blow-up $\mathrm{Bl}_{Y'} \mathrm{Bl}_Y \mathbb{P}^3$, the boundary component $(E, E_{\geq 0})$ of $(\mathrm{Bl}_Y \mathbb{P}^3, \widetilde{\mathcal{P}})$ is blown-up at the point labelled $(p, c)$, which is not a positive geometry for the same reason as in Example 4.5.1 (i.e. because we are blowing-up a point in the interior of an edge). In particular, the sequential blow-up is not a positive geometry. ◇

**Example 4.5.7.** We now give an example illustrating (iii). In other words, we show that $\mathcal{B}$ being a building set is insufficient, and the well-adapted condition on $\mathcal{B}$ is necessary for the blow-up to be a positive geometry.

Consider a four-dimensional polytope $\mathcal{P}$, which is a pyramid over a triangular prism. Its set of vertices and a Schlegel diagram are given in Figure 4.9.

Let $F_1, F_2$, and $H$ be the linear spans of the vertices (considered as vectors in $\mathbb{R}^4$ via $\mathbb{R}^4 \cong \{X_0 \neq 0\} \subset \mathbb{P}^4(\mathbb{R})$) labelled by $\{2, 4, 5\}, \{1, 4, 6\}$, and $\{0, 1, 2, 3, 4\}$, respectively. Note that $\mathcal{P} \cap \mathbb{P}H$ recovers the square pyramid. Consider the set $\mathcal{B} = \{F_1, F_2\}$, which is a building set; indeed, the codimensions of $F_1$ and $F_2$ add to the codimension of $F_1 \cap F_2$.
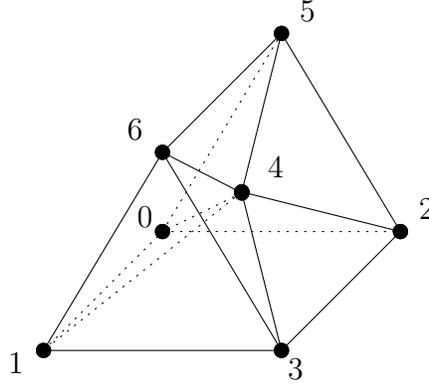
Figure 4.9: A Schlegel diagram of $\mathcal{P}$. Vertex 4 is the top of the pyramid. The ordered vertices can be given coordinates $\{0000, 1000, 0100, 1100, 0010, 0001, 1001\}$ in $\mathbb{R}^4 \subset \mathbb{P}^4(\mathbb{R})$.

However, $\mathcal{B}_H = \{F_1 \cap \mathbb{P}H, F_2 \cap \mathbb{P}H\}$ recovers the two opposite lines of Example 4.5.6, which we have seen is not a building set in $\mathbb{P}H$, and does not result in a positive geometry. ◇

### 4.5.4 A further example

We provide one non-polytopal example where the blow-up of a boundary stratum results in a positive geometry.

**Example 4.5.8.** Consider the semialgebraic set

$$S := \left\{[X : Y : Z : W] \in \mathbb{P}^3(\mathbb{R}) \mid W > 0 \ \text{ and } \ Z, W - Z, W - Y, Y - X^2 \geq 0\right\} \subset \mathbb{P}^3(\mathbb{R}).$$

Consider the affine chart $U := \{W \neq 0\} \subset \mathbb{P}^3$ with dehomogenized coordinates $x = \frac{X}{W}, y = \frac{Y}{W}$, and $z = \frac{Z}{W}$. Our set $S$ is a full-dimensional semialgebraic subset of $U(\mathbb{R}) \cong \mathbb{A}^3(\mathbb{R})$, cut out by the following inequalities:

$$0 \leq z \leq 1 \ \text{ and } \ x^2 \leq y \leq 1.$$

The pair $(\mathbb{P}^3, S)$ is a positive geometry with canonical form $\Omega = \Omega(\mathbb{P}^3, S)$ given by

$$\Omega\big|_U = \frac{2}{(y - x^2)(y - 1)z(z - 1)} dx \, dy \, dz.$$

The subvariety $Y := V(Z, YW - X^2) \subset \mathbb{P}^3$ is the intersection of the boundary components $\{Z = 0\}$ and $\{YW = X^2\}$ of $(\mathbb{P}^3, S)$. Figure 4.10 depicts the two boundary components and their intersection in our affine chart $U = \{W \neq 0\}$. The blow-up $\mathrm{Bl}_Y \mathbb{P}^3$ has exceptional divisor $E = \mathbb{P}(N_{Y|\mathbb{P}^3}) \cong \mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$, which is not isomorphic to a product of two projective spaces. Computations in affine-local coordinates show that $\left(\mathrm{Bl}_Y \mathbb{P}^3, \widetilde{S}\right)$ is a positive geometry with canonical form $\pi^*\Omega$.

◇

Figure 4.10: Left: The boundary components $\{z = 0\}$ in green and $\{y = x^2\}$ in orange, and their intersection $Y = V(z, y - x^2)$ in blue. Right: The exceptional divisor $E$ of $\mathrm{Bl}_Y \mathbb{P}^3$ in grey and its nonnegative part $E_{\geq 0}$ in blue.

**Remark 4.5.9.** Example 4.5.8 indicates a possible future direction of study: one can investigate whether blow-ups of *polypols* (see [Koh+25]) are positive geometries.

In Part I, the study of Grasstopes and exterior cyclic polytopes demonstrates how the topology and convex geometry of amplituhedron-like objects depend sensitively on positivity. This last chapter has taken a more algebra-geometric approach, and demonstrates that positive geometries in the sense of [ABL17] do not in general have log resolutions. We emphasize that the foundations of the field of positive geometry are still being written. Indeed, the work of [BD25] builds a new framework in which log resolutions are automatic. We are excited by this new perspective.

# Part II

# Classical algebraic geometry

# Chapter 5

# The Chow–Lam Form

This chapter is based on [PS25], which is joint work with Bernd Sturmfels. The content is largely the same, with changes to the notation for Schubert varieties.

The goal of this chapter is to develop the theory and practise of Chow–Lam forms, using that of Chow forms as a guide. In this, we transition from subvarieties of projective space to subvarieties of Grassmannians.

The name "Chow–Lam form" recognizes work of Thomas Lam [Lam16b; Lam16a] at the interface of combinatorics and particle physics. Lam focuses on the case when $\mathcal{V}$ is a positroid variety, namely a boundary component of the positive Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$. Concretely, each of these varieties is defined by the vanishing of certain collections of Plücker coordinates. Lam refers to $\mathcal{CL}_{\mathcal{V}}$ as the *universal amplituhedron variety* [Lam16b, Section 18.1], and he discusses universal projections via twistor coordinates. We now review Lam's *degree-three example* from [Lam16b, Section 19.4].

**Example 5.0.1** (A positroid variety)**.** Fix $k=2, n=9, r=7$ and the positroid variety

$$\mathcal{V} \; = \; V(q_{12}, q_{13}, q_{23}, q_{45}, q_{67}, q_{89}) \; \subset \; \mathrm{Gr}(2, 9).$$

This is the positroid $\beta = (3, 2, 2, 2)$ in the notation of Section 5.3. Then $\mathcal{CL}_{\mathcal{V}}$ is a hypersurface in $\mathrm{Gr}(4, 9)$. It consists of all $4 \times 9$-matrices $X$ whose columns, viewed as points in $\mathbb{P}^3$, satisfy: some line in the plane 123 meets the lines 45, 67 and 89. This happens if and only if the three lines intersect the plane in collinear points. We can write these intersection points explicitly in Plücker coordinates. For example, intersection of 123 and 45 is the point $q_{2345}x_1 - q_{1345}x_2 + q_{1245}x_3$, where $x_i$ is the $i$th column of $X$. Therefore, the Chow–Lam form is

$$\mathrm{CL}_{\mathcal{V}} = \mathrm{d}et \begin{bmatrix} q_{2345} & -q_{1345} & q_{1245} \\ q_{2367} & -q_{1367} & q_{1267} \\ q_{2389} & -q_{1389} & q_{1289} \end{bmatrix} = q_{1234}q_{1237}q_{5689} + q_{1234}q_{1236}q_{5789} - q_{1235}q_{1236}q_{4789} - q_{1235}q_{1237}q_{4689}.$$

Projections of $\mathcal{V}$ into $\mathrm{Gr}(2, 7)$ satisfy equations in twistor coordinates, by Corollary 5.1.7. $\diamond$

We start in Section 5.1 with a review of coordinate systems on Grassmannians and basics on Chow forms, such as the Intersection Formula and the Projection Formula. In Section

5.2 we develop the corresponding theory of Chow–Lam forms. Building on the work of Lam [Lam16a], we give a criterion for $\mathcal{CL}_\mathcal{V}$ to have codimension one, and we present a formula for the degree of $\mathrm{CL}_\mathcal{V}$. In Section 5.3 we turn to varieties given by matroids and positroids. We compute their Chow–Lam degrees and Chow–Lam forms in some interesting cases, such as Theorem 5.3.6. In Section 5.4 we introduce the Hurwitz-Lam form, which governs non-transversal intersections of complementary dimension. This generalizes [Stu17] and can be used to compute branch loci.

## 5.1 Coordinates and Chow forms

This chapter develops tools for computing with subvarieties in Grassmannians. To this end, it is important for us to be precise about the coordinates to be used. We distinguish four different coordinate systems to represent a linear subspace $L \subset \mathbb{C}^n$, corresponding to a point in $\mathrm{Gr}(k, n)$. What follows is consistent with the conventions adopted in [DS95; Stu92; Stu17].

If $L$ is given as the kernel of an $(n - k) \times n$-matrix then the entries of that matrix are called the *primal Stiefel coordinates* and its maximal minors are the *primal Plücker coordinates*, denoted $p_{i_1 i_2 \cdots i_{n-k}}$. If $L$ is given to us as the row space of a $k \times n$-matrix then the entries of that matrix are called the *dual Stiefel coordinates* and its maximal minors are the *dual Plücker coordinates*, denoted $q_{j_1 j_2 \cdots j_k}$. After complementing indices, primal and dual Plücker coordinates agree up to multiplication by $(-1)^{j_1 + \cdots + j_k}$. This sign agrees with the sign $\epsilon(J, J^c)$ from Section 3.1, where $J = (j_1, ..., j_k)$. For example, the ten Plücker coordinates on $\mathrm{Gr}(3, 5)$ are

$$
\begin{array}{lllllllllll}
\text{Primal} & p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45}, \\
\text{Dual} & q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123}.
\end{array}
\tag{5.1}
$$

In geometric applications, $L$ represents a projective subspace of dimension $k - 1$ in $\mathbb{P}^{n-1}$. In the primal perspective, $L$ is given as the intersection of hyperplanes. In the dual perspective, $L$ is the span of points. Which of these is preferable depends on whether $k$ or $n - k$ is smaller.

**Remark 5.1.1.** Plücker coordinates are always alternating with respect to permuting indices. For instance, in (5.1) we have $-q_{245} = q_{254} = q_{425} = -q_{452} = -q_{524} = q_{542}$.

Let $\mathcal{V}$ be an irreducible variety of dimension $d$ in $\mathbb{P}^{n-1}$. We now define the Chow form of $\mathcal{V}$. Let $\mathcal{C}_\mathcal{V}$ be the subvariety of $\mathrm{Gr}(n - d - 1, n)$ whose points are subspaces $L$ such that $L \cap \mathcal{V} \neq \emptyset$ in $\mathbb{P}^{n-1}$. Then $\mathcal{C}_\mathcal{V}$ has codimension one. Since each Grassmannian has Picard group $\mathbb{Z}$, the hypersurface $\mathcal{C}_\mathcal{V}$ is the zero set of a single polynomial in Plücker coordinates. This polynomial is denoted by $\mathrm{C}_\mathcal{V}$ and called the *Chow form* of $\mathcal{V}$. It is unique up to the Plücker relations. We will show in Corollary 5.1.5 that the degree of $\mathrm{C}_\mathcal{V}$ in Plücker coordinates equals the degree of $\mathcal{V}$ in $\mathbb{P}^{n-1}$.

The Chow form $\mathrm{C}_\mathcal{V}$ can be written in either primal Plücker coordinates, dual Plücker coordinates, primal Stiefel coordinates, or dual Stiefel coordinates. All four variants are useful, depending on the context. We illustrate this for the rational normal curve in $\mathbb{P}^4$.

**Example 5.1.2** ($d = 1, n = 5$). Let $\mathcal{V}$ be the curve $(1 : t : t^2 : t^3 : t^4)$ in $\mathbb{P}^4$. As in [DS95, Section 1.2], its Chow form in primal Plücker coordinates is the determinant of the *Bézout matrix*:

$$\mathrm{C}_{\mathcal{V}} \;=\; \det \begin{bmatrix} p_{12} & p_{13} & p_{14} & p_{15} \\ p_{13} & p_{14} + p_{23} & p_{15} + p_{24} & p_{25} \\ p_{14} & p_{15} + p_{24} & p_{25} + p_{34} & p_{35} \\ p_{15} & p_{25} & p_{35} & p_{45} \end{bmatrix}. \tag{5.2}$$

Passing to primal Stiefel coordinates $p_{ij} = a_i b_j - a_j b_i$, we obtain the *Sylvester resultant*

$$\mathrm{C}_{\mathcal{V}} \;=\; \mathrm{Res}_t\big(a_1 + a_2 t + a_3 t^2 + a_4 t^3 + a_5 t^4, \; b_1 + b_2 t + b_3 t^2 + b_4 t^3 + b_5 t^4\big).$$

For the formula in dual Plücker coordinates, replace each $p_{ij}$ with the $\pm q_{klm}$ below it in (5.1). Replacing the $q_{klm}$ with the $3 \times 3$ minors of a $3 \times 5$ matrix, we obtain the formula for $\mathrm{C}_{\mathcal{V}}$ in dual Stiefel coordinates. This is a polynomial of degree 12 in 15 unknowns. It characterizes triples of binary quartics whose linear span contains the fourth power of some linear form. ◇

The determinantal expressions in Examples 5.1.2 and Example 1.3.1 (the twisted cubic) have generalizations; indeed, an analogue to the Bézout formula exists for arbitrary curves in $\mathbb{P}^{n-1}$. This is explained in [ES03, Section 4]. In their article [ES03], Eisenbud and Schreyer present a method for computing determinantal formulas for Chow forms. This rests on syzygies for Ulrich sheaves on $\mathcal{V}$. We are optimistic that this generalizes to Chow–Lam forms in the Grassmannian setting.

But first we review basics on Chow forms, following the exposition in [DS95]. We present the formulas for intersections and projections of projective varieties in terms of their Chow forms. We begin with intersections. The following result is found in [DS95, Proposition 2.1].

**Proposition 5.1.3** (Intersection Formula). *Suppose* $\dim(\mathcal{V}) = d$, *and let* $L$ *and* $M$ *be linear subspaces of* $\mathbb{P}^{n-1}$ *such that* $\mathrm{codim}(L \cap M) = \mathrm{codim}(L) + \mathrm{codim}(M) = d + 1$. *Then*

$$\mathrm{C}_{\mathcal{V} \cap L}(M) \;=\; \mathrm{C}_{\mathcal{V}}(L \cap M). \tag{5.3}$$

To use this formula in practice, we need to express the Plücker coordinates of $L \cap M$ via those of $L$ and $M$. Let $\ell$ and $m$ be the dual Plücker coordinates of $L$ and $M$ respectively. Then the dual Plücker coordinates of $L \cap M$ are given by the exterior product of $\ell$ and $m$:

$$p \;=\; \ell \wedge m. \tag{5.4}$$

For example, let $n = 6$ and $d = 2$ and suppose that $\mathrm{codim}(L) = 1$ and $\mathrm{codim}(M) = 2$. Then $\ell = (\ell_1, \ell_2, \ldots, \ell_6)$, $m = (m_{12}, m_{13}, \ldots, m_{56})$, and the 20 coordinates of (5.4) are as follows:

$$p_{ijk} \;=\; \ell_i m_{jk} - \ell_j m_{ik} + \ell_k m_{ij} \qquad \text{for} \quad 1 \le i < j < k \le 6. \tag{5.5}$$

We illustrate this formula in a concrete scenario of interest in elimination theory [Stu92].

**Example 5.1.4** (Veronese surface). Fix the surface $\mathcal{V} = \{(1 : x : y : x^2 : xy : y^2)\}$ in $\mathbb{P}^5$. Its Chow form $\mathrm{C}_{\mathcal{V}}$ is the resultant of three ternary quadrics. Explicitly, $\mathrm{C}_{\mathcal{V}}$ is a polynomial of degree 4 in primal Plücker coordinates $p_{ijk}$ on $\mathrm{Gr}(3,6)$. See [Stu92, Section 2.2] for the formula. The expansion of $\mathrm{C}_{\mathcal{V}}$ into primal Stiefel coordinates has 21894 terms; see [Stu02, eqn (4.5)].

Let $L$ be a hyperplane in $\mathbb{P}^5$, with coordinates $\ell_i$. The curve $\mathcal{V} \cap L$ is the Veronese embedding of the conic $V(\ell_1 + \ell_2 x + \ell_3 y + \ell_4 x^2 + \ell_5 xy + \ell_6 y^2) \subset \mathbb{P}^2$. By substituting (5.5) into $\mathrm{C}_{\mathcal{V}}$, we obtain the Chow form $\mathrm{C}_{\mathcal{V} \cap L}$ of this curve in primal Plücker coordinates $m_{ij}$.

Next let $\mathrm{codim}(L) = 2$, with Plücker coordinates $\ell_{ij}$. Then $\mathcal{V} \cap L$ consists of four points in $\mathbb{P}^5$. The Chow form $\mathrm{C}_{\mathcal{V} \cap L}(m_1, \ldots, m_6)$ is a quartic that factors into four linear forms. $\diamond$

**Corollary 5.1.5.** *The degree of the Chow form $\mathrm{C}_{\mathcal{V}}$ equals the degree of the variety $\mathcal{V}$.*

*Proof.* Let $\mathrm{codim}(L) = d = \dim(\mathcal{V})$ in Proposition 5.1.3. By (5.3), the Chow forms $\mathrm{C}_{\mathcal{V}}$ and $\mathrm{C}_{\mathcal{V} \cap L}$ have the same degree in their Plücker coordinates. We claim that this degree is $\delta = \deg(\mathcal{V})$. This holds because $\mathcal{V} \cap L$ is a finite set $\{u^{(i)} = (u_1^{(i)} : \cdots : u_n^{(i)}) : i = 1, 2, \ldots, \delta\}$. Its Chow form equals $\mathrm{C}_{\mathcal{V} \cap L}(m) = \prod_{i=1}^{\delta} (u_1^{(i)} m_1 + \cdots + u_n^{(i)} m_n)$. This has degree $\delta$ in $m$. $\square$

We now turn to projections $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$. These are given by $r \times n$ matrices $Z = (z_{ij})$. Let $\mathcal{V} \subset \mathbb{P}^{n-1}$ be a variety of dimension $d \leq r - 2$. We write $Z(\mathcal{V})$ for the closure of the image of $\mathcal{V}$ in $\mathbb{P}^{r-1}$. For general $Z$, the variety $Z(\mathcal{V})$ has dimension $d$ and degree $\delta = \deg(\mathcal{V})$ in $\mathbb{P}^{r-1}$. The hypersurface $\mathcal{C}_{Z(\mathcal{V})}$ lives in $\mathrm{Gr}(r-d-1, r)$. We shall write its defining polynomial $\mathrm{C}_{Z(\mathcal{V})}$ in terms of dual Stiefel coordinates. We represent an element of $\mathrm{Gr}(r-d-1, r)$ as the column span of an $r \times (r-d-1)$ matrix $Y$ with unknown entries. The concatenation $[Z \,|\, Y]$ is a matrix with $r$ rows and $n + r - d - 1$ columns. We now review Definition 1.1.3 of *twistor coordinates*. For any sequence $1 \leq i_1 < i_2 < \cdots < i_{d+1} \leq n$, there is a twistor coordinate $[Z \,|\, Y]_{i_1 i_2 \cdots i_{d+1}}$. This is the $r \times r$ subdeterminant of $[Z \,|\, Y]$ given by the $r - d - 1$ columns of $Y$ and the $d + 1$ columns of $Z$ indexed by $i_1, i_2, \ldots, i_{d+1}$.

The twistor coordinate $[Z \,|\, Y]_{i_1 i_2 \cdots i_{d+1}}$ is a linear form in the maximal minors of the matrix $Y$. These are dual Plücker coordinates on $\mathrm{Gr}(r-d-1, r)$. They are preferred when $r - d$ is small. However, if $d$ is small then it is better to use primal Plücker coordinates. We find these by multiplying $Z$ on the left with a matrix of primal Stiefel coordinates.

**Proposition 5.1.6** (Projection Formula). *The Chow form $\mathrm{C}_{Z(\mathcal{V})}$ in dual Stiefel coordinates equals the Chow form $\mathrm{C}_{\mathcal{V}}$ with primal Plücker coordinates replaced with twistor coordinates:*

$$p_{i_1 i_2 \cdots i_{d+1}} \;=\; [Z \,|\, Y]_{i_1 i_2 \cdots i_{d+1}} \quad \text{for } 1 \leq i_1 < i_2 < \cdots < i_{d+1} \leq n. \qquad (5.6)$$

*Proof.* This is a reinterpretation of the formula in [DS95, Section 2.2] which was derived for the projection from a point. In more geometric terms, our formula can be written as follows:

$$\mathrm{C}_{Z(\mathcal{V})}(Y) \;=\; \mathrm{C}_{\mathcal{V}}(Z^{-1}(Y)).$$

Here $Y$ is a subspace of dimension $r - d - 2$ in $\mathbb{P}^{r-1}$. Its preimage $Z^{-1}(Y)$ is a subspace of dimension $n - d - 2$ in $\mathbb{P}^{n-1}$. We have $Y \cap Z(\mathcal{V}) \neq \emptyset$ if and only if $Z^{-1}(Y) \cap \mathcal{V} \neq \emptyset$. $\square$

The case of most interest in the projection formula is $r = d + 2$, when $Z(\mathcal{V})$ is a hypersurface of degree $\delta$ in $\mathbb{P}^{r-1}$. Here $Y = (y_1, y_2, \ldots, y_r)^T$ is the column vector of coordinates on $\mathbb{P}^{r-1}$. Each of the $\binom{n}{r-1}$ twistor coordinates $[\, Z \, | \, Y \,]_{i_1 i_2 \cdots i_{r-1}}$ is a linear form in $y_1, y_2, \ldots, y_r$.

**Corollary 5.1.7.** *The equation of any hypersurface obtained by projecting $\mathcal{V} \subset \mathbb{P}^{n-1}$ into $\mathbb{P}^{r-1}$ is read off from the Chow form $\mathrm{C}_{\mathcal{V}}$ by replacing Plücker coordinates with linear forms via (5.6).*

**Example 5.1.8.** Rational quartic curves in $\mathbb{P}^2$ are images of the curve $\mathcal{V}$ in Example 5.1.2 under projections $Z : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$. To compute these plane curves, we consider the $3 \times 6$ matrix

$$[\, Z \, | \, Y \,] \;=\; \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & y_1 \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & y_2 \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & y_3 \end{bmatrix}.$$

The equation of the plane quartic curve $Z(\mathcal{V})$ is obtained from the Bézout determinant in (5.2) by replacing $p_{ij}$ with the $3 \times 3$ minor of this $3 \times 6$ matrix having column indices $i, j, 6$.

We can similarly project the Veronese surface (Example 5.1.4) into $\mathbb{P}^3$ with a $4 \times 6$ matrix $Z$. The resulting rational quartics are known as *Roman surfaces* or *Steiner surfaces*. Algebraically, we substitute twistor coordinates into the Chow form in [Stu92, Section 2.2].

$\diamond$

The classical theory of Chow forms extends naturally to a hierarchy of higher Chow forms, which characterize linear spaces that are tangent to $\mathcal{V}$. These are also known as coisotropic hypersurfaces. Their degrees are the polar degrees of $\mathcal{V}$, as shown by Kohn in [Koh21]. On the far end of the hierarchy is the dual variety, which characterizes hyperplanes tangent to $\mathcal{V}$. On the near end, right next to the Chow form, is the Hurwitz form, which we now review.

Fix $\mathcal{V} \subset \mathbb{P}^{n-1}$ of dimension $d$ and degree $\delta$. For $L \in \mathrm{Gr}(n-d, n)$ generic, the intersection $\mathcal{V} \cap L$ consists of $\delta$ distinct points. Let $\mathcal{H}_{\mathcal{V}}$ be the subvariety of $\mathrm{Gr}(n-d, n)$ consisting of all $L$ where this fails. Geometrically, such $L$ are tangent to $\mathcal{V}$. If $\delta \geq 2$ then $\mathcal{H}_{\mathcal{V}}$ is a hypersurface in $\mathrm{Gr}(n-d, n)$. This hypersurface was studied in [Stu17]. Its defining equation $\mathrm{H}_{\mathcal{V}}$ is the *Hurwitz form* of $\mathcal{V}$. We close this section by deriving the Hurwitz analog to Corollary 5.1.7.

**Theorem 5.1.9.** *The equation of the branch locus of any projection of $\mathcal{V} \subset \mathbb{P}^{n-1}$ into $\mathbb{P}^d$ is read off from the Hurwitz form $\mathrm{H}_{\mathcal{V}}$ by replacing Plücker coordinates with linear forms via (5.6).*

*Proof.* The branch locus consists of all points $y$ such that the fiber $Z^{-1}(y)$ is tangent to $\mathcal{V}$ at some point. This happens if and only if the Hurwitz form $\mathrm{H}_{\mathcal{V}}$ vanishes at the subspace $Z^{-1}(y)$. Evaluating the Hurwitz form in primal Plücker coordinates $p$ at any such fiber translates into the algebraic operation of replacing $p$ with twistor coordinates, by Proposition 5.1.6. $\qquad\square$

**Example 5.1.10** (Branch curve of the Veronese)**.** Let $Z$ be a general $3 \times 6$ matrix, defining a projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$, and let $\mathcal{V}$ be the Veronese surface in Example 5.1.4. The Hurwitz form $\mathrm{H}_{\mathcal{V}}$ has degree six in the Plücker coordinates. Its expansion in primal Stiefel coordinates is

the *tact invariant* of two conics. For the explicit formula in primal Plücker coordinates $p_{ij}$ see [Stu17, Example 2.7]. The branch curve in $\mathbb{P}^2$ has degree six. Its equation is obtained from $H_\mathcal{V}$ by replacing $p_{ij}$ with the $3 \times 3$ minor indexed by $i, j, 7$ in the $3 \times 7$ matrix $[\, Z \,|\, Y \,]$. $\diamond$

## 5.2 From projective space to the Grassmannian

In the previous section, we encoded a subvariety $\mathcal{V}$ in $\mathbb{P}^{n-1} = \mathrm{Gr}(1, n)$ by a single equation $C_\mathcal{V}$. We here replace the ambient projective space with an arbitrary Grassmannian $\mathrm{Gr}(k, n)$. Now the degree of $\mathcal{V}$ is no longer an integer but a cohomology class $[\mathcal{V}]$. Recall that $H^*(\mathrm{Gr}(k, n), \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\binom{n}{k}}$, with basis given as follows.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$, and let $E_i := \mathrm{span}(e_1, \ldots, e_i)$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ fitting inside a $k \times (n - k)$ box, define the Schubert variety $\Omega_\lambda$ to be

$$\Omega_\lambda = \{L \in \mathrm{Gr}(k, n) \ : \ \dim L \cap E_{n-k+\lambda_i-i} \geq i\}. \tag{5.7}$$

For convenience we may sometimes write our partitions in the form $m^{a_m} \ldots 1^{a_1}$ where $a_j$ is the number of parts of size $j$. For example, $\lambda = (3, 3, 2, 2, 2, 2, 0)$ would be written as $3^2 2^4$. For more on Schubert varieties, see [FP98].

The Schubert variety $\Omega_\lambda$ is a closed irreducible subvariety of codimension $\sum_i \lambda_i$. In particular, $\Omega_\varnothing$ is the class of $\mathrm{Gr}(k, n)$ and $\Omega_{(n-k)^k}$ is the class of a point.

The Schubert classes $[\Omega_\lambda]$ form a $\mathbb{Z}$-basis of $H^*(\mathrm{Gr}(k, n), \mathbb{Z})$. For any subvariety $\mathcal{V}$, its cohomology class has a unique expansion into Schubert classes:

$$[\mathcal{V}] \ = \ \sum_\lambda \delta_\lambda(\mathcal{V}) \cdot [\Omega_\lambda]. \tag{5.8}$$

Each coefficient $\delta_\lambda(\mathcal{V})$ is a nonnegative integer, which is zero unless $|\lambda| = \mathrm{codim}(\mathcal{V})$.

There is a natural involution on the set of partitions within a box of dimension $k \times (n-k)$. Namely, the complement of $\lambda$ is the partition

$$\lambda^c \ = \ \big\{\, n - k - \lambda_k, n - k - \lambda_{k-1}, \, \ldots, \, n - k - \lambda_1 \,\big\}.$$

The Schubert varieties $\Omega_\lambda$ and $\Omega_{\lambda^c}$ have complementary dimensions. For a general matrix $g \in \mathrm{GL}(n, \mathbb{C})$, and any partition $\nu$ that satisfies $\dim \Omega_\nu = \dim \Omega_\lambda = \mathrm{codim} \, \Omega_{\lambda^c}$, we have

$$g \, \Omega_\nu \ \cap \ \Omega_{\lambda^c} \ = \ \begin{cases} \text{a point} & \text{if } \lambda = \nu, \\ \emptyset & \text{otherwise.} \end{cases}$$

This gives the following method to compute the cohomology class (5.8) from the ideal of $\mathcal{V}$.

**Proposition 5.2.1.** *The coefficient $\delta_\lambda(\mathcal{V})$ in the class $[\mathcal{V}]$ of the subvariety $\mathcal{V} \subset \mathrm{Gr}(k, n)$ equals the number of points in the intersection $g \, \mathcal{V} \cap \Omega_{\lambda^c}$ for a general matrix $g \in \mathrm{GL}(n, \mathbb{C})$.*

We can thus compute the numbers $\delta_\lambda(\mathcal{V})$ either symbolically (using Gröbner bases) or numerically (using numerical algebraic geometry). The latter approach uses *Schubert witness sets* from homotopy continuation. These were introduced by Sottile in [Sot20, Section 4].

We now come to our main topic, namely the Chow–Lam form. Recall that $\mathcal{V} \subset \mathrm{Gr}(k, n)$ was assumed to have dimension $k(r - k) - 1$ for some $r \in \{k + 1, \ldots, n\}$. We define $\mathcal{CL}_\mathcal{V}$ to be the subvariety of $\mathrm{Gr}(k + n - r, n)$ which consists of all points $P$ such that the inclusion $Q \subseteq P$ holds for some $Q \in \mathcal{V}$. We call $\mathcal{CL}_\mathcal{V}$ the *Chow–Lam locus* of the given variety $\mathcal{V}$.

**Lemma 5.2.2.** *If $\mathcal{V}$ is irreducible then $\mathcal{CL}_\mathcal{V}$ is a proper irreducible subvariety of $\mathrm{Gr}(k + n - r, n)$.*

*Proof.* We expect $\mathcal{CL}_\mathcal{V}$ to be a hypersurface in $\mathrm{Gr}(k+n-r, n)$. Indeed, there are $k(r-k)-1$ degrees of freedom in fixing a point $Q \in \mathcal{V}$. After fixing $Q$, we choose $P$. The variety of all subspaces $P \in \mathrm{Gr}(k + n - r, n)$ that contain $Q$ is a Grassmannian $\mathrm{Gr}(n - r, n - k)$. So, there are $(n - r)(r - k) = \dim(\mathrm{Gr}(n - r, n - k))$ degrees of freedom for choosing $P$. Our construction parametrizes an irreducible incidence variety whose dimension is the sum

$$k(r-k) - 1 \; + \; (n-r)(r-k) \; = \; (r-k)(k+n-r) - 1 \; = \; \dim(\mathrm{Gr}(k+n-r, n)) - 1. \quad (5.9)$$

The incidence variety parametrizes all pairs $(Q, P)$ as above. It is irreducible and maps onto the Chow–Lam locus $\mathcal{CL}_\mathcal{V}$. This shows $\mathcal{CL}_\mathcal{V}$ is irreducible of dimension at most (5.9). $\square$

The *Chow–Lam form* of $\mathcal{V}$ is the polynomial $\mathrm{CL}_\mathcal{V}$ that defines the Chow–Lam locus $\mathcal{CL}_\mathcal{V}$, provided this has codimension one. Otherwise, $\mathcal{V}$ is *degenerate* and we set $\mathrm{CL}_\mathcal{V} = 1$. This definition is analogous to the definition of a degenerate dual variety and discriminant (cf. [GKZ94]). For any projective variety, the dual variety is expected to be a hypersurface, and its defining polynomial is the discriminant. However, it can happen that the dual variety has codimension $\geq 2$, in which case the discriminant is 1. We note that $\mathrm{CL}_\mathcal{V}$ is unique up to scaling and modulo the Plücker relations for $\mathrm{Gr}(k + n - r, n)$. We can write it either in Plücker coordinates (primal or dual) or in Stiefel coordinates (primal or dual).

**Remark 5.2.3.** In our definition of the Chow–Lam locus it is assumed that the variety $\mathcal{V}$ has dimension $k(r-k)-1$. This is, of course, a restrictive hypothesis. Our rationale for this is that we would like $\mathcal{CL}_\mathcal{V}$ to have codimension one. Giving up this hypothesis would take us to the general setting of higher Chow–Lam loci, which concerns tangencies between $\mathcal{V}$ and arbitrary subGrassmannians $\mathrm{Gr}(k, L)$ of $\mathrm{Gr}(k, n)$. This is discussed at the end of Section 5.4.

The Chow–Lam form is named for Thomas Lam, who studies universal projections of positroid varieties in [Lam16b, Chapter 18]. We will establish the relationship between universal projections and the Chow–Lam form in Proposition 5.2.10. Lam also computes cohomology classes of projections in [Lam16a, Proposition 3.5]. This is fundamental for Theorem 5.2.5 below.

We next explore the issue of degeneracy for Schubert varieties in a small Grassmannian.

**Example 5.2.4** ($k = 2, n = 5, r = 4$). Let $\mathcal{V}$ be a variety of dimension 3 in the 6-dimensional Grassmannian $\mathrm{Gr}(2,5)$. Here, $k + n - r = 3$, so the Chow–Lam locus $\mathcal{CL}_\mathcal{V}$ is a subvariety of $\mathrm{Gr}(3,5)$. We shall use dual Plücker coordinates $q$ on $\mathrm{Gr}(2,5)$ and primal Plücker coordinates $p$ on $\mathrm{Gr}(3,5)$. The inclusion $Q \subset P$ is expressed algebraically by the matrix equation

$$
\begin{bmatrix}
0 & p_{12} & p_{13} & p_{14} & p_{15} \\
-p_{12} & 0 & p_{23} & p_{24} & p_{25} \\
-p_{13} & -p_{23} & 0 & p_{34} & p_{35} \\
-p_{14} & -p_{24} & -p_{34} & 0 & p_{45} \\
-p_{15} & -p_{25} & -p_{35} & -p_{45} & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & q_{12} & q_{13} & q_{14} & q_{15} \\
-q_{12} & 0 & q_{23} & q_{24} & q_{25} \\
-q_{13} & -q_{23} & 0 & q_{34} & q_{35} \\
-q_{14} & -q_{24} & -q_{34} & 0 & q_{45} \\
-q_{15} & -q_{25} & -q_{35} & -q_{45} & 0
\end{bmatrix}
= 0. \qquad (5.10)
$$

This is a system of 25 bilinear equations in $(p, q)$. We augment this by the equations in $q$ that define $\mathcal{V}$, we saturate with respect to the $q$-variables, and we finally eliminate all $q$-variables. The resulting ideal in the ten $p$-variables defines the Chow–Lam locus $\mathcal{CL}_\mathcal{V} \subset \mathrm{Gr}(3,5)$.

First suppose that $\mathcal{V}$ is defined by three general linear forms in $p$. Then $\mathcal{CL}_\mathcal{V}$ has codimension one, and the Chow–Lam form $\mathrm{CL}_\mathcal{V}$ has degree two. For a concrete example, take $\mathcal{V} = V(q_{12} + q_{13}, q_{24} + q_{25}, q_{23} + q_{35})$. The threefold $\mathcal{V}$ has degree five in Plücker space $\mathbb{P}^9$. It parametrizes all lines in $\mathbb{P}^4$ that meet three given planes in $\mathbb{P}^4$. The algorithm above yields

$$
\mathrm{CL}_\mathcal{V} = p_{14}\,(p_{24} - p_{25} - p_{34} + p_{35}) + (p_{12} - p_{14} + p_{15})\,p_{45}.
$$

Next, we consider the two Schubert varieties of dimension 3 in $\mathrm{Gr}(2,5)$. They are

$$
\Omega_{21} = V(q_{15}, q_{25}, q_{35}, q_{45}, q_{34}) \quad \text{and} \quad \Omega_3 = V(q_{23}, q_{24}, q_{34}, q_{25}, q_{35}, q_{45}).
$$

The first Schubert threefold is non-degenerate and its Chow–Lam form equals $\mathrm{CL}_{\Omega_{21}} = p_{12}$. The second Schubert threefold $\Omega_3$ is found to be degenerate. Our algorithm reveals that

$$
\mathcal{CL}_{\Omega_3} = V(p_{12}, p_{13}, p_{14}, p_{15}).
$$

This Chow–Lam locus has codimension two in $\mathrm{Gr}(3,5)$, and hence $\mathrm{CL}_{\Omega_3} = 1$. $\diamond$

We now come to the first result of this section. It will explain the findings in Example 5.2.4. Let $\alpha = \{n - r + 1, n - r, \ldots, n - r\}$. We take the convention that our parts weakly decrease in size. Then any partition with total size equal to the codimension of $\mathcal{V}$ satisfies $\alpha_1 \geq n - r + 1$, and $\alpha$ is the unique partition for which we have equality. We set $\alpha(\mathcal{V}) := \delta_\alpha(\mathcal{V})$ and call this the *Chow–Lam degree* of the given variety $\mathcal{V}$. Note that the Chow–Lam degrees of the three varieties in Example 5.2.4 are $\alpha(\mathcal{V}) = 2$, $\alpha(\Omega_{21}) = 1$, and $\alpha(\Omega_3) = 0$.

**Theorem 5.2.5.** *Let $\mathcal{V}$ be a subvariety of dimension $k(r - k) - 1$ in $\mathrm{Gr}(k,n)$. The Chow–Lam form $\mathrm{CL}_\mathcal{V}$ is a polynomial of degree $\alpha(\mathcal{V})$ in the Plücker coordinates on $\mathrm{Gr}(k+n-r, n)$. In particular, $\mathcal{V}$ is degenerate if and only if its Chow–Lam degree $\alpha(\mathcal{V})$ is zero.*

This formula for the degree of $\mathrm{CL}_\mathcal{V}$ generalizes Corollary 5.1.5. Indeed, if $k = 1$ then our variety $\mathcal{V}$ has dimension $d = r - 2$ in $\mathbb{P}^{n-1} = \mathrm{Gr}(1, n)$, and its cohomology class (5.8) is

$$[\mathcal{V}] = \deg(\mathcal{V}) \cdot [\Omega_d].$$

Here, the Chow–Lam degree is just the classical degree, i.e. $\alpha(\mathcal{V}) = \deg(\mathcal{V})$. The argument that led to Corollary 5.1.5 can be generalized to $k \geq 2$. Our proof of Theorem 5.2.5 will be based on this. We begin with the generalization of the intersection formula in Proposition 5.1.3.

For a subspace $L$ of dimension $\ell$ in $\mathbb{C}^n$, we introduce the relative Grassmannian

$$\mathrm{Gr}(k, L) = \big\{ P \in \mathrm{Gr}(k, n) \,|\, P \subseteq L \big\} \quad \simeq \quad \mathrm{Gr}(k, \ell).$$

With this notation, the definition of the Chow–Lam locus can be rewritten as follows:

$$\mathcal{CL}_\mathcal{V} = \big\{ L \in \mathrm{Gr}(k + n - r, n) : \mathcal{V} \cap \mathrm{Gr}(k, L) \neq \emptyset \big\}. \tag{5.11}$$

**Lemma 5.2.6** (Intersection Formula). *Let $L$ and $M$ be linear subspaces of $\mathbb{P}^{n-1}$ such that $\mathrm{codim}(L \cap M) = \mathrm{codim}(L) + \mathrm{codim}(M) = r - k$. Then*

$$\mathrm{CL}_{\mathcal{V} \cap \mathrm{Gr}(k, L)}(M) = \mathrm{CL}_\mathcal{V}(L \cap M). \tag{5.12}$$

*Proof.* This follows from (5.11) and the identity $\mathrm{Gr}(k, L \cap M) = \mathrm{Gr}(k, L) \cap \mathrm{Gr}(k, M)$. $\square$

**Corollary 5.2.7.** *The Chow–Lam forms $\mathrm{CL}_\mathcal{V}$ and $\mathrm{CL}_{\mathcal{V} \cap \mathrm{Gr}(k, L)}$ have the same degree, when written in their respective Plücker coordinates.*

The following lemma is based on a standard argument in Schubert calculus.

**Lemma 5.2.8.** *Let $L$ be a general subspace of dimension $\rho$ in $\mathbb{C}^n$. The intersection $\Omega_\lambda \cap \mathrm{Gr}(k, L)$ is non-empty if and only if $\lambda_1 + \rho < n - k + 1$. In this case, its class is a Schubert class, namely*

$$[\Omega_\lambda \cap \mathrm{Gr}(k, L)] = [\Omega_{(\lambda_1 + \rho, \,\ldots,\, \lambda_k + \rho)}].$$

**Corollary 5.2.9.** *Using notation as above, we have $\alpha(\mathcal{V}) = \alpha(\mathcal{V} \cap \mathrm{Gr}(k, L))$.*

*Proof of Theorem 5.2.5.* By Corollaries 5.2.7 and 5.2.9, the assertion follows by induction on $n - r$. It suffices to prove the assertion for the base case $n - r = 0$, where $\mathcal{V} = \mathcal{CL}_\mathcal{V}$ is a hypersurface in $\mathrm{Gr}(k, n)$. Note that $[\mathcal{V}] = \alpha(\mathcal{V}) \cdot [\Omega_1]$. This $[\Omega_1]$ is the class of a hyperplane section of $\mathrm{Gr}(k, n)$ inside its Plücker embedding. Hence $\mathcal{V}$ is defined by a polynomial of degree $\alpha(\mathcal{V})$ in the $\binom{n}{k}$ Plücker coordinates, which is unique modulo Plücker relations. Moreover, this polynomial equals the Chow–Lam form $\mathrm{CL}_\mathcal{V}$, since $\mathcal{V} = \mathcal{CL}_\mathcal{V}$. $\square$

We now come to the projection formula for $k \geq 2$, which is the direct generalization of Proposition 5.1.6. Let $Z$ be a general $r \times n$ matrix as in Section 2. Then $Z$ defines a projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$ and this induces a rational map $\mathrm{Gr}(k, n) \dashrightarrow \mathrm{Gr}(k, r)$. We write $Z(\mathcal{V})$ for the closure in $\mathrm{Gr}(k, r)$ of the image of the variety $\mathcal{V} \subset \mathrm{Gr}(k, n)$ under the map induced by $Z$.

**Proposition 5.2.10** (Projection Formula). *The Chow–Lam form* $\mathrm{CL}_{Z(\mathcal{V})}$ *is obtained from the Chow–Lam form* $\mathrm{CL}_{\mathcal{V}}$ *by replacing the primal Plücker coordinates with twistor coordinates:*

$$p_{i_1 i_2 \cdots i_{r-k}} \;=\; [\, Z \,|\, Y \,]_{i_1 i_2 \cdots i_{r-k}} \quad \text{for } 1 \le i_1 < i_2 < \cdots < i_{r-k} \le n. \qquad (5.13)$$

*This expresses* $\mathrm{CL}_{Z(\mathcal{V})}$ *in dual Stiefel coordinates on* $\mathrm{Gr}(k, r)$, *given by the* $r \times k$ *matrix* $Y$.

*Proof.* We identify the $r \times k$ matrix $Y$ with its column span, which is a $(k-1)$-dimensional subspace in $\mathbb{P}^{r-1}$, or a $k$-dimensional linear subspace in $\mathbb{C}^r$. The inverse image $Z^{-1}(Y)$ is a projective subspace of dimension $n - r + k - 1$ in $\mathbb{P}^{n-1}$, or a linear subspace of dimension $n - r + k$ in $\mathbb{C}^n$. The subspace $Y$ is a point in $\mathcal{CL}_{Z(\mathcal{V})}$ if and only if $Z^{-1}(Y)$ lies in $\mathcal{CL}_{\mathcal{V}}$. $\square$

**Corollary 5.2.11.** *Any hypersurface obtained by projecting* $\mathcal{V} \subset \mathrm{Gr}(k, n)$ *into* $\mathrm{Gr}(k, r)$ *is read off from* $\mathrm{CL}_{\mathcal{V}}$ *by replacing primal Plücker coordinates with twistor coordinates via* (5.13).

**Example 5.2.12** (Genus 6 geometry). Fix $k = 2$ and $n = 5$. We start with the case $r = 4$. Let $\mathcal{V}$ be the subvariety of $\mathrm{Gr}(2, 5)$ defined by two general linear forms and one general quadric in the 10 dual Plücker coordinates $q_{ij}$. Then $\mathcal{V}$ is a *Fano threefold* of genus 6, see [Log12]. This threefold has degree 10 in the ambient space $\mathbb{P}^9$. Its Chow–Lam degree is $\alpha(\mathcal{V}) = 4$. We compute $\mathrm{CL}_{\mathcal{V}}$ as in Example 5.2.4, i.e. we augment the ideal of $\mathcal{V}$ by the bilinear equations in (5.10) and then eliminate the $q$-variables after saturation. The result is a quartic in primal Plücker coordinates $p_{ij}$ on $\mathrm{Gr}(3, 5)$. This is the Chow–Lam form $\mathrm{CL}_{\mathcal{V}}$ of our Fano threefold $\mathcal{V}$.

Consider the projection $\mathrm{Gr}(2, 5) \dashrightarrow \mathrm{Gr}(2, 4)$ given by a general $4 \times 5$ matrix $Z$. The image $Z(\mathcal{V})$ is a hypersurface in $\mathrm{Gr}(2, 4)$. Its defining polynomial $\mathrm{CL}_{Z(\mathcal{V})}$ is obtained by setting $p_{ij} = [\, Z \,|\, Y \,]_{ij}$. These twistor coordinates are the $4 \times 4$ minors of the $4 \times 7$ matrix $[\, Z \,|\, Y \,]$ which use the last two columns. Hence $\mathrm{CL}_{Z(\mathcal{V})}$ has degree 4 in the $2 \times 2$ minors of $Y$.

We next assume that $r = 3$ and $\mathcal{V} \subset \mathrm{Gr}(2, 5)$ is cut out by four general linear forms and one general quadric in the $q_{ij}$. Here $\mathcal{V}$ is a *canonical curve* of genus 6 and degree 10 in the $\mathbb{P}^5$ defined by the four linear forms; see [Gei+23, Lemma 4.1]. We have $\alpha(\mathcal{V}) = 10$, so the Chow–Lam form $\mathrm{CL}_{\mathcal{V}}$ is a polynomial of degree 10 in the coordinates $(p_1, \ldots, p_5)$ on $\mathrm{Gr}(4, 5) \simeq (\mathbb{P}^4)^\vee$.

Consider the projection $\mathrm{Gr}(2, 5) \dashrightarrow \mathrm{Gr}(2, 3)$ given by a general $3 \times 5$ matrix $Z$. The image $Z(\mathcal{V})$ is a singular curve of degree 10 in the dual projective plane $\mathrm{Gr}(2, 3) = (\mathbb{P}^2)^\vee$, with coordinates $(y_1, y_2, y_3)$. We find its equation by the substitution into twistor coordinates; see (1.2). $\diamond$

The join of two projective varieties is an important operation in algebraic geometry. The construction was characterized at the level of Chow forms in [DS95, Section 4.1]. The definition of the join generalizes to subvarieties of Grassmannians and their Chow–Lam forms.

Let $\mathcal{V}$ and $\mathcal{W}$ be disjoint subvarieties of $\mathrm{Gr}(k, n)$. We define the *embedded join* of $\mathcal{V}$ and $\mathcal{W}$ to be the union

$$J(\mathcal{V}, \mathcal{W}) \quad := \quad \bigcup_{V \in \mathcal{V}, W \in \mathcal{W}} \mathrm{Gr}(k, V + W).$$

This is a subvariety in $\mathrm{Gr}(k,n)$. The dimension of $J(\mathcal{V},\mathcal{W})$ equals $\dim\mathcal{V}+\dim\mathcal{W}+k^2$, provided this number does not exceed $k(n-k)$. The following result is dual to Lemma 5.2.6.

**Proposition 5.2.13.** *Fix $\mathcal{V}\subset\mathrm{Gr}(k,n)$ of dimension $k(r-k)-1$, and consider general subspaces $L,M\subset\mathbb{C}^n$ with $\mathrm{codim}(L+M)=r-k$. We have the equality of Chow–Lam forms*

$$\mathrm{CL}_{J(\mathcal{V},\mathrm{Gr}(k,L))}(M) \;=\; \mathrm{CL}_{\mathcal{V}}(L+M). \tag{5.14}$$

The numerology of dimensions matches since $\dim J(\mathcal{V},\mathrm{Gr}(k,L))=k(r+\dim(L)-k)-1$. We omit the proof, and instead close with an example to illustrate the embedded join.

**Example 5.2.14** (Joining a curve). Let $L\in\mathrm{Gr}(3,6)$ and fix a curve $\mathcal{V}\subset\mathrm{Gr}(2,6)$. Then the join of $\mathcal{V}$ and $\mathrm{Gr}(2,L)\simeq\mathbb{P}^2$ is a hypersurface in $\mathrm{Gr}(2,6)$. Its equation is obtained by writing $\mathrm{CL}_{\mathcal{V}}$ in dual Plücker coordinates for $L+M$. For instance, if $\mathcal{V}$ is a genus 8 canonical curve [Gei+23, Section 4] then $\alpha(\mathcal{V})=14$, and therefore $\mathrm{CL}_{J(\mathcal{V},\mathrm{Gr}(k,L))}$ has degree 14. $\qquad\diamond$

## 5.3 Matroids and positroids

In this section we turn to subvarieties of $\mathrm{Gr}(k,n)$ that are defined by the vanishing of Plücker coordinates. For a point $\xi\in\mathrm{Gr}(k,n)$ in dual Plücker coordinates, the associated *matroid* $M=M_\xi$ is the set of all indices $I\in\binom{[n]}{k}$ such that $\xi_I=0$. Thus $M$ is a matroid of rank $k$ on $[n]$, given by its list of *nonbases*. The nonbases are the indices of dual Plücker coordinates that are zero. The matroid $M$ is a *positroid* if $M=M_\xi$ for some $\xi$ with dual Plücker coordinates all nonnegative real numbers.

Conversely, for a matroid $M$ of rank $k$ on $[n]$, the *realization space* is the constructible set $\{\xi\in\mathrm{Gr}(k,n):M_\xi=M\}$. The Zariski closure of the realization space in the Grassmannian $\mathrm{Gr}(k,n)$ is denoted $\mathcal{V}_M$ and called the *matroid variety* of $M$. Positroids are of special interest at the interface of combinatorics and physics, and one uses the term *positroid variety* for $\mathcal{V}_M$ if $M$ happens to be a positroid. Positroid varieties behave much better than general matroid varieties; as shown in [KLS13]. We seek to compute the Chow–Lam forms of these varieties. The images $Z(\mathcal{V}_M)$ of positroid varieties $\mathcal{V}_M$ under projections $Z:\mathrm{Gr}(k,n)\dashrightarrow\mathrm{Gr}(k,r)$ are known as *amplituhedron varieties*. Lam's work in [Lam16b] aims at computing the equations defining $Z(\mathcal{V}_M)$ in twistor coordinates, using the Projection Formula for the Chow–Lam form.

**Example 5.3.1** ($k=2,n=6,r=5$). The matroid $M=\{12,34,56\}$ is a positroid. Its positroid variety $\mathcal{V}_M=V(\xi_{12},\xi_{34},\xi_{56})$ is a subvariety of dimension $k(r-k)-1=5$ in the 8-dimensional Grassmannian $\mathrm{Gr}(2,6)$. This variety has Chow–Lam degree $\alpha(\mathcal{V}_M)=2$. Its Chow–Lam form is the condition for the three lines $\overline{12}$, $\overline{34}$ and $\overline{56}$ in $\mathbb{P}^2$ to be concurrent:

$$\mathrm{CL}_{\mathcal{V}_M} \;=\; p_{123}p_{456} \,-\, p_{124}p_{356}. \tag{5.15}$$

The derivation of this formula is similar to (but easier than) that in Example 5.0.1. The amplituhedron variety $Z(\mathcal{V}_M)$ is a hypersurface in $\mathrm{Gr}(2,5)$. Its equation is found by replacing the 20 $p_{ijk}$ with the $5\times5$ minors of the $5\times7$ matrix $[\,Z\,|\,Y\,]$ that use the last two columns. $\quad\diamond$

**Remark 5.3.2** (Schubert matroids). Among the matroid varieties $\mathcal{V}_M$ are the Schubert varieties $\Omega_\lambda$. The associated matroids $M_\lambda$ are very special. Let $I = (\lambda_1 + n - k + 1, \lambda_2 + n - k + 2, \ldots, \lambda_k + n)$. The bases of $M_\lambda$ are the index sets $J$ such that $J \geq I$. Oh [Oh11] characterizes positroids in terms of such *Schubert matroids*. Namely, positroid varieties $\mathcal{V}_M$ are intersections of cyclically shifted Schubert varieties.

Every matroid variety $\mathcal{V}_M$ is naturally stratified into toric varieties. The torus $T = (\mathbb{C}^*)^n$ acts on the Grassmannian $\mathrm{Gr}(k,n)$. Given any point $\xi \in \mathrm{Gr}(k,n)$, we write $\mathcal{T}_\xi$ for the Zariski closure of the orbit $T\xi$ in $\mathrm{Gr}(k,n)$. This is a toric variety of dimension $\leq n-1$, and the dimension equals $n-1$ when $\xi$ is generic. Up to a multiplicative change of coordinates, the toric varieties $\mathcal{T}_\xi$ depend only on the matroid $M = M_\xi$, and we therefore write $\mathcal{T}_M = \mathcal{T}_\xi$.

We call $\mathcal{T}_M$ the toric variety associated with the matroid $M$. The corresponding polytope is the *matroid polytope* of $M$. For example, if $\xi$ is a generic point in $\mathrm{Gr}(k,n)$ then $M$ is the uniform matroid, and the matroid polytope is the hypersimplex $\Delta(k,n)$. Here, $\mathcal{T}_M$ is the $(n-1)$-dimensional toric variety parametrized by all $\binom{n}{k}$ squarefree monomials of degree $k$ in $n$ unknowns. In Chapter 7 we study the Chow–Lam forms of the toric varieties $\mathcal{T}_M$.

**Example 5.3.3** (Hypersimplex). We fix a general $2 \times 6$ matrix $(a_{ij})$ with row span $\xi \in \mathrm{Gr}(2,6)$. The toric variety $\mathcal{T}_\xi$ has dimension 5 and degree 26 in $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$. Its prime ideal is generated by 30 quadrics like $\xi_{13}\xi_{24}\,q_{12}q_{34} - \xi_{12}\xi_{34}\,q_{13}q_{24}$. The Chow–Lam form is

$$\begin{aligned}
\mathrm{CL}_{\mathcal{T}_\xi} &= (\xi_{14}\xi_{26}\xi_{35} + \xi_{15}\xi_{24}\xi_{36} - \xi_{16}\xi_{24}\xi_{35})\,q_{123}q_{456} - \xi_{13}\xi_{25}\xi_{46}\,q_{124}q_{356} \\
&\quad + \xi_{12}\xi_{35}\xi_{46}\,q_{134}q_{256} - \xi_{12}\xi_{34}\xi_{56}\,q_{135}q_{246} + \xi_{13}\xi_{24}\xi_{56}\,q_{125}q_{346}.
\end{aligned} \tag{5.16}$$

Modulo the Plücker ideal, $\mathrm{CL}_{\mathcal{T}_\xi}$ is the unique relation among the $3 \times 3$ minors of the matrix

$$\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 & a_{14}x_4 & a_{15}x_5 & a_{16}x_6 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 & a_{24}x_4 & a_{25}x_5 & a_{26}x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix}.$$

Indeed, this matrix is a parametric representation of $\mathcal{CL}_{\mathcal{T}_\xi}$, with $x_i$ and $y_j$ as parameters. This is a subvariety of dimension 8 in the Grassmannian $\mathrm{Gr}(3,6)$. Note that (5.15) is obtained from (5.16) by setting $\xi_{12} = \xi_{34} = \xi_{56} = 0$ and $\xi_{ij} = 1$ for all other $i, j$. This reflects the fact that $\mathcal{T}_M = \mathcal{V}_M$ for the positroid $M$ in Example 5.3.1, and that $\mathrm{CL}_{\mathcal{T}_\xi}$ specializes to $\mathrm{CL}_{\mathcal{T}_M}$. $\diamond$

We next focus on matroids of rank $k = 2$. In this case, every matroid $M$ is a positroid after relabeling. We assume that $M$ is loopless, i.e. for every $i$ there exists $j$ such that $\xi_{ij} \neq 0$. With this mild hypothesis, every matroid is encoded by a partition $\beta = (\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t)$ of the integer $n$ into $t \geq 2$ parts. The non-bases are the pairs that are contained in one of the $t$ blocks $\{1, \ldots, \beta_1\}, \{\beta_1 + 1, \ldots, \beta_2\}, \ldots, \{\beta_{t-1} + 1, \ldots, \beta_t\}$ in the corresponding set partition of $[n]$. From now on, we identify rank 2 matroids with rank 2 positroids, we encode them by partitions $\beta$, and we write $\mathcal{V}_\beta$ for the associated positroid variety in $\mathrm{Gr}(2,n)$. We note that $\mathrm{codim}(\mathcal{V}_\beta) = n - t$ and $r = \frac{n+t+1}{2}$. In particular, $\mathcal{V}_{(1,1,1,\ldots,1)} = \mathrm{Gr}(2,n)$ and $\mathcal{V}_{(2,1,\ldots,1)} = \Omega_1$.

**Proposition 5.3.4.** *Let $\beta$ be a partition of $n$ with $t$ parts, and fix the univariate polynomial*

$$f(x) \;=\; \prod_{i=1}^{t} \sum_{j=1}^{\beta_i} x^{j-1}.$$

*Suppose that $n - t$ is odd. Then the Chow–Lam degree of the positroid variety $\mathcal{V}_\beta$ is the coefficient of the monomial $x^{(n-t-1)/2}$ in $f(x)$ minus the coefficient of $x^{(n-t-3)/2}$ in $f(x)$.*

*Proof.* We represent Schubert classes by symmetric polynomials. The variety $\mathcal{V}_\beta$ is the intersection of Schubert varieties of the form $\Omega_{\beta_i-1}$ for $i = 1, 2, \ldots, t$. The class $[\Omega_{\beta_i-1}]$ can be represented by the complete homogeneous symmetric polynomial $h_{\beta_i-1}(x, y)$ of degree $\beta_i - 1$ in two variables $x$ and $y$. Hence $[\mathcal{V}_\beta]$ is the product of these symmetric polynomials:

$$[\mathcal{V}_\beta] \;=\; h_{\beta_1-1}(x, y)\, h_{\beta_2-1}(x, y) \,\cdots\, h_{\beta_t-1}(x, y).$$

We write this product as a sum of Schubert classes $x^i y^{n-t-i} + x^{i+1} y^{n-t-i-1} + \cdots + x^{n-t-i} y^i$ for $i = 0, 1, \ldots, (n-t-1)/2$. The Chow–Lam degree is the coefficient of the middle Schubert class $[\Omega_{n-r+1,n-r}]$ in this decomposition. We dehomogenize it by setting $y = 1$. Thereafter we conclude by observing that $x^{(n-t-1)/2}$ occurs once in each class $[\Omega_\lambda]$ of codimension $n - t$, whereas $x^{(n-t-3)/2}$ occurs once in each class except $[\Omega_{n-r+1,n-r}]$. $\qquad\square$

**Example 5.3.5.** For $n \leq 8$ all Chow–Lam degrees $\alpha(\mathcal{V}_\beta)$ are 0, 1 or 2. For every integer $n$ between 9 and 30, the maximal value of $\alpha(\mathcal{V}_\beta)$ is attained by a unique partition $\beta$. In the following table we list all triples $n, \alpha(\mathcal{V}_\beta), \beta$ that give the maxima for $n$ in that range:

$$9, 3, (3222) \qquad 10, 5, (22222) \qquad 11, 5, (222221) \qquad 12, 6, (33222) \qquad 13, 9, (322222)$$
$$14, 14, (2222222) \quad 15, 14, (22222221) \quad 16, 19, (3322222) \quad 17, 28, (32222222) \quad 18, 42, (222222222)$$
$$19, 43, (33322222) \qquad 20, 62, (332222222) \qquad 21, 90, (3222222222) \qquad 22, 132, (2222222222)$$
$$23, 145, (3332222222) \quad 24, 207, (33222222222) \quad 25, 297, (32222222222) \quad 26, 429, (2222222222222)$$
$$27, 497, (333222222222) \quad 28, 704, (3322222222222) \quad 29, 1001, (32222222222222) \quad 30, 1430, (222222222222222)$$

The first entry says that the 9-dimensional positroid variety $\mathcal{V}_{(3222)} \subset \mathrm{Gr}(2, 9)$ has Chow–Lam degree 3. The cubic Chow–Lam form $\mathrm{CL}_{\mathcal{V}_\beta}$ was computed in Example 5.0.1. One thing we notice in our table is the appearance of Catalan numbers whenever $n \equiv 2 \bmod 4$. This calls for an explanation. We shall provide one, along with a new tool for computing $\mathrm{CL}_{\mathcal{V}_\beta}$. $\quad\diamond$

The degree of the Grassmannian $\mathrm{Gr}(2, s) \subset \mathbb{P}^{\binom{s}{2}-1}$ is the Catalan number

$$C_{s-2} \;=\; \frac{1}{s-1}\binom{2s-4}{s-2}.$$

Hence the Chow form $\mathrm{C}_{\mathrm{Gr}(2,s)}$ has degree $C_{s-2}$ in Plücker coordinates, and it has degree $(2s-3) \cdot C_{s-2}$ in primal Stiefel coordinates. This Chow form is the tool we promised above.

**Theorem 5.3.6.** *Fix the partition $\beta = (2, 2, 2, \ldots, 2)$ of $n = 2t$ where $t = 2s - 3$ is odd. The Chow–Lam degree of the positroid variety $\mathcal{V}_\beta$ is the Catalan number $C_{s-1} = \frac{n}{s} C_{s-2}$. The Chow–Lam form $\mathrm{CL}_{\mathcal{V}_\beta}$ has degree $s\, C_{s-1} = n\, C_{s-2}$ in dual Stiefel coordinates, given by an $s \times n$ matrix $X = [\, x_1\; x_2\; \cdots\; x_n \,]$. We obtain $\mathrm{CL}_{\mathcal{V}_\beta}$ from the Chow form of the Grassmannian $\mathrm{Gr}(2, s)$ by evaluating the primal Stiefel coordinates at the columns of the $\binom{s}{2} \times t$ matrix*

$$\tilde{X} \;=\; \begin{bmatrix} x_1 \wedge x_2 & x_3 \wedge x_4 & x_5 \wedge x_6 & \cdots & x_{n-1} \wedge x_n \end{bmatrix}.$$

This theorem explains the Catalan numbers in Example 5.3.5, since these are the degrees of the Grassmannians $\mathrm{Gr}(2, s)$. Before proving Theorem 5.3.6, we go over the case $s = 4$.

**Example 5.3.7** (Five lines admitting a transversal). Let $\beta = (2, 2, 2, 2, 2)$ and fix a $4 \times 10$ matrix $X = [\, x_1\; x_2\; \cdots\; x_9\; x_0 \,]$. Its columns $x_i$ are viewed as points in $\mathbb{P}^3$. We are interested in the following condition on $X$: there exists a line $L$ which intersects the five lines $\overline{x_1 x_2}$, $\overline{x_3 x_4}$, $\overline{x_5 x_6}$, $\overline{x_7 x_8}$, $\overline{x_9 x_0}$. This codimension 1 condition is given by the Chow–Lam form of $\mathcal{V}_\beta$:

$$\mathrm{CL}_{\mathcal{V}_\beta} \;=\; \det \begin{bmatrix} 0 & q_{1234} & q_{1256} & q_{1278} & q_{1290} \\ q_{1234} & 0 & q_{3456} & q_{3478} & q_{3490} \\ q_{1256} & q_{3456} & 0 & q_{5678} & q_{5690} \\ q_{1278} & q_{3478} & q_{5678} & 0 & q_{7890} \\ q_{1290} & q_{3490} & q_{5690} & q_{7890} & 0 \end{bmatrix}. \tag{5.17}$$

Here the $q_{ijkl}$ are maximal minors of $X$, so they are dual Plücker coordinates on $\mathrm{Gr}(4, 10)$. The expansion into the entries of $X$ has degree 20 and it is the sum of $18\,268\,320$ monomials.

The formula (5.17) appears in [Stu93, Theorem 3.4.7], and it is derived as follows. Write the dual and primal Plücker coordinates of the five lines as the columns of the two $6 \times 5$ matrices

$$\begin{aligned} \tilde{X} &\;=\; \begin{bmatrix} x_1 \wedge x_2 & x_3 \wedge x_4 & x_5 \wedge x_6 & x_7 \wedge x_8 & x_9 \wedge x_0 \end{bmatrix} \\ (\tilde{X})^* &\;=\; \begin{bmatrix} (x_1 \wedge x_2)^* & (x_3 \wedge x_4)^* & (x_5 \wedge x_6)^* & (x_7 \wedge x_8)^* & (x_9 \wedge x_0)^* \end{bmatrix}. \end{aligned}$$

The symmetric $5 \times 5$ matrix $(\tilde{X})^T \cdot (\tilde{X})^*$ is precisely the matrix in (5.17). The left kernel of $\tilde{X}$ is spanned by the vector $(L_{12}, \ldots, L_{34})$ of primal Plücker coordinates of the transversal line $L$. Using Cramer's rule, we write each $L_{ij}$ as a $5 \times 5$ minor of the matrix $\tilde{X}$. We compute

$$\mathrm{CL}_{\mathcal{V}_\beta} \;=\; \mathrm{C}_{\mathrm{Gr}(2,4)} \;=\; L_{12}L_{34} - L_{13}L_{24} + L_{14}L_{23} \;=\; \frac{1}{2}\, \det\!\big((\tilde{X})^T \cdot (\tilde{X})^*\big). \tag{5.18}$$

It remains to be seen that this is the Chow–Lam form of $\mathcal{V}_\beta = V(q_{12}, q_{34}, q_{56}, q_{78}, q_{90})$. The matrix $X$ represents a point in $\mathcal{CL}_{\mathcal{V}_\beta}$ if and only if its row space has a subspace $Q$ that lies in $\mathcal{V}_\beta$. This means that we can find a $2 \times 4$ matrix $T$ such that the $2 \times 10$ matrix $Q = TX$ has its five minors $\{2i - 1, 2i\}$ vanish. Then $L = \wedge_2 T$ is precisely the transversal line above. $\diamond$

*Proof of Theorem 5.3.6.* The matrix $X$ represents a point in $\mathcal{CL}_{\mathcal{V}_\beta}$ if and only if there exists a $2 \times s$ matrix $T$ of rank 2 such that the matrix $TX$ has its consecutive minors $\{2i-1, 2i\}$ vanish. We view the entries of $T$ as dual Stiefel coordinates of a line in $\mathbb{P}^{s-1}$, with dual Plücker coordinates $L = \wedge_2 T$. We view the columns $x_{2i-1} \wedge x_{2i}$ of $\tilde{X}$ as hyperplanes in the ambient space $\mathbb{P}^{\binom{s}{2}-1}$ of the Grassmannian $\mathrm{Gr}(2,s)$. Our condition says that these $t$ hyperplanes intersect $\mathrm{Gr}(2,s)$ in a common point $L$. Since $t = 2s - 3 = \dim(\mathrm{Gr}(2,s)) + 1$, this imposes one constraint on $X$. The polynomial in the entries of $X$ that defines this hypersurface is the Chow form of $\mathrm{Gr}(2,s)$, now evaluated at the $t$ hyperplanes $x_{2i-1} \wedge x_{2i}$. $\qquad\square$

**Example 5.3.8** (Chow form of $\mathrm{Gr}(2,5)$)**.** We present a formula for $\mathrm{C}_{\mathrm{Gr}(2,5)}$ in dual Stiefel coordinates, given by three skew-symmetric $5 \times 5$ matrices $U, V, W$. We ask whether some linear combination $P = xU + yV + zW$ has rank 2. The five Plücker relations $f_i = p_{jk}p_{lm} - p_{jl}p_{km} + p_{jm}p_{kl}$ are quadrics in $x, y, z$. Let $f_6$ be the derivative of the Jacobian determinant of $f_1, f_2, f_3$ with respect to $z$. Following [Stu02, eqn (4.5)], we form the $6 \times 6$ matrix of coefficients from the ternary quadrics $f_1, f_2, \ldots, f_6$. Its determinant has degrees $(5, 5, 6)$ in $(U, V, W)$, and it equals $\mathrm{C}_{\mathrm{Gr}(2,5)}$ times $w_{45}$. Note that $\mathrm{C}_{\mathrm{Gr}(2,5)}$ has degree 5 in Plücker coordinates.

By rewriting this Chow form in primal Plücker coordinates, as above, we obtain a formula for the Chow–Lam form $\mathrm{CL}_{\mathcal{V}_\beta}$, where $\beta = (2, 2, 2, 2, 2, 2, 2)$. This has degree 14 in Plücker coordinates on $\mathrm{Gr}(5, 14)$. At present we do not know any determinantal formula like (5.17).

It remains a challenge to find practical tools for computing the Chow forms of $\mathrm{Gr}(2,n)$ when $n \geq 6$. A formula for the power $(\mathrm{C}_{\mathrm{Gr}(2,n)})^{n-2}$ in dual Stiefel coordinates appears in [Apr+24, Section 3]. This is based on resonance varieties for Koszul modules. It would be desirable to implement and test this formula. Can the exponent $n-2$ be removed in this construction? $\qquad\diamond$

**Remark 5.3.9.** Theorem 5.3.6 generalizes to a positroid $M$ on $n = kt$ elements in rank $k \geq 3$. It has precisely $t$ nonbases which are pairwise disjoint. We assume that $t = (s-k)k+1$ for some integer $s > k$. The Chow–Lam degree of the positroid variety $\mathcal{V}_M \subset \mathrm{Gr}(k,n)$ equals

$$\alpha(\mathcal{V}_M) \;=\; \frac{n}{s} \cdot \deg \mathrm{Gr}(k,s),$$

and we can derive $\mathrm{CL}_{\mathcal{V}_M}$ from $\mathrm{C}_{\mathrm{Gr}(k,s)}$ as above. The coordinates are the entries of an $s \times n$ matrix $X$ and an $\binom{s}{k} \times t$ matrix $\widetilde{X}$, obtained by wedging the $k$-clusters of columns in $X$.

For example, if $k = 3, s = 6$ then $t = 10$, $n = 30$, and $\mathcal{V}_M$ is a variety of dimension 71 in $\mathrm{Gr}(3, 30)$. Its Chow–Lam degree equals $210 = \frac{n}{s} \cdot 42 = \frac{n}{s} \cdot \deg \mathrm{Gr}(3,6)$. The $24 \times 30$ matrix $Z$ maps $\mathcal{V}_M$ to a hypersurface in $\mathrm{Gr}(3, 24)$, with equation found from $\mathrm{CL}_{\mathcal{V}_M}$ by Corollary 5.1.7.

In this section we presented techniques for studying the Chow–Lam forms of matroid varieties $\mathcal{V}_M$. A matrix $X$ of dual Stiefel coordinates represents a point in $\mathcal{CL}_{\mathcal{V}_M}$ if there exists a matrix $T$ such that $TX$ realizes the matroid $M$. We can compute $\mathrm{CL}_{\mathcal{V}_M}$ by eliminating $T$ from the equations $TX \in \mathcal{V}_M$. It can be preferable to perform the elimination directly with

dual Plücker coordinates, i.e. with the maximal minors of $X$. This approach is recommended when $\mathcal{V}_M$ is given parametrically. We conclude the section with such an example.

**Example 5.3.10** (A rank 3 matroid)**.** Fix the matroid $M = \{126, 135, 234, 456\}$. Here $k = 3$, $n = 6$, and $r = 5$, so $\mathcal{V}_M$ has codimension 4 in $\mathrm{Gr}(3, 6)$, and it projects to a hypersurface in $\mathrm{Gr}(3, 5)$. Its Chow–Lam locus $\mathcal{CL}_{\mathcal{V}_M}$ is a subvariety of $\mathrm{Gr}(4, 6)$, given by the parametrization

$$
X \;=\; \begin{bmatrix} x_1 & 0 & 0 & 0 & -x_5 & x_6 \\ 0 & x_2 & 0 & x_4 & 0 & -x_6 \\ 0 & 0 & x_3 & -x_4 & x_5 & 0 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix}.
$$

Indeed, the first three rows parametrize $\mathcal{V}_M$. The maximal minors satisfy the cubic relation

$$
\mathrm{CL}_{\mathcal{V}_M} \;=\; q_{1234}q_{1356}q_{2456} + q_{1235}q_{1246}q_{3456} - q_{1235}q_{1346}q_{2456}.
$$

This form is unique modulo the Plücker relations, and is easily found by elimination. ◇

## 5.4 Hurwitz-Lam forms and beyond

The purpose of this section is to generalize the Hurwitz form [Stu17] for subvarieties of projective space to subvarieties of Grassmannians. Theorem 5.1.9 extends to that setting and provides a tool for computing the branch loci of projections of subvarieties of Grassmannians. At the end of this section, we generalize even further by introducing higher Chow–Lam forms.

Let $\mathcal{V}$ be a subvariety of $\mathrm{Gr}(k, n)$ of dimension $k(r - k)$ for some $r \in \{k+1, \ldots, n\}$. Fix a general point $P \in \mathrm{Gr}(k + n - r, n)$. This represents a linear subspace in $\mathbb{C}^n$, and we consider the Grassmannian $\mathrm{Gr}(k, P)$, which parametrizes all linear subspaces $Q \in \mathrm{Gr}(k, n)$ such that $Q \subseteq P$. We are interested in the intersection of $\mathcal{V}$ with the subGrassmannian $\mathrm{Gr}(k, P)$.

**Lemma 5.4.1.** *The intersection*

$$
\mathcal{V} \;\cap\; \mathrm{Gr}(k, P) \tag{5.19}
$$

*is a finite set of points. The number of points is the coefficient $\delta_\lambda(\mathcal{V})$ in (5.8), where $\lambda = \{n - r, n - r, \ldots, n - r\}$. We call $\delta_\lambda(\mathcal{V})$ the* Hurwitz-Lam degree *of the variety $\mathcal{V}$.*

*Proof.* The proof is a dimension argument, like that leading to Lemma 5.2.2. The identification of the specific Schubert class $\Omega_\lambda$ is analogous to that in the derivation of Theorem 5.2.5. □

We define the *Hurwitz-Lam locus* of the given variety $\mathcal{V}$ as follows:

$$
\mathcal{HL}_{\mathcal{V}} \;=\; \big\{\, P \in \mathrm{Gr}(k + n - r, n) \,:\, \text{the intersection (5.19) is not transverse} \big\}. \tag{5.20}
$$

If the Hurwitz-Lam degree is at least two then $\mathcal{HL}_{\mathcal{V}}$ is a hypersurface in $\mathrm{Gr}(k + n - r, n)$. The polynomial in Plücker coordinates that defines this hypersurface is the *Hurwitz-Lam form*, denoted $\mathrm{HL}_{\mathcal{V}}$. If the Hurwitz-Lam locus is not a hypersurface then we set $\mathrm{HL}_{\mathcal{V}} = 1$.

The Hurwitz-Lam form $\mathrm{HL}_\mathcal{V}$ of a subvariety $\mathcal{V} \subset \mathrm{Gr}(k,n)$ can be used to describe the branch loci of projections to smaller Grassmannians. The set-up is as in Proposition 5.1.6.

**Theorem 5.4.2** (Computing Branch Loci). *Let $\mathcal{V}$ be a variety in $\mathrm{Gr}(k,n)$ whose Hurwitz-Lam degree $\delta_I(\mathcal{V})$ is at least two. A general linear projection $Z$ maps $\mathcal{V}$ onto $\mathrm{Gr}(k,r)$. The branch locus is a hypersurface in $\mathrm{Gr}(k,r)$, and its equation is obtained from the Hurwitz-Lam form $\mathrm{HL}_\mathcal{V}$ by replacing the primal Plücker coordinates with twistor coordinates as in (5.13).*

*Proof.* A point $Y \in \mathrm{Gr}(k,r)$ lies in the branch locus of the map $Z$ from $\mathcal{V}$ if and only if the fiber $Z^{-1}(Y)$ intersects $\mathcal{V}$ non-transversally. This happens if and only if $Z^{-1}(Y)$ is in $\mathcal{HL}_\mathcal{V}$. That the branch locus is codimension one follows from the Zariski-Nagata Purity Theorem. This ensures that $\mathcal{HL}_\mathcal{V}$ has codimension one. We obtain the equation of the branch locus from $\mathrm{HL}_\mathcal{V}$ by the same arguments as in Proposition 5.2.10. For $k = 1$ see Theorem 5.1.9. $\square$

We illustrate Theorem 5.4.2 with a case study that is motivated by amplituhedra and other Grassmann polytopes; see Chapter 2. We set $k = 2, n = 8$ and we let $\mathcal{V} = \mathcal{V}_\beta = V(q_{12}, q_{34}, q_{56}, q_{78})$ be the positroid variety in $\mathrm{Gr}(2,8)$ given by the partition $\beta = (2,2,2,2)$. We have $\mathrm{codim}(\mathcal{V}) = 4$ and $\dim(\mathcal{V}) = 8$. The Hurwitz-Lam degree of $\mathcal{V}$ is two. This means that a general linear projection $Z : \mathrm{Gr}(2,8) \dashrightarrow \mathrm{Gr}(2,6)$ induces a 2-to-1 map from $\mathcal{V}$ onto $\mathrm{Gr}(2,6)$. The branch locus of this map is a hypersurface in $\mathrm{Gr}(2,6)$. The next proposition gives its equation.

**Proposition 5.4.3.** *The Hurwitz-Lam form of the positroid variety $\mathcal{V} \subset \mathrm{Gr}(2,8)$ equals*

$$\mathrm{HL}_\mathcal{V} \;=\; \det \begin{bmatrix} 0 & q_{1234} & q_{1256} & q_{1278} \\ q_{1234} & 0 & q_{3456} & q_{3478} \\ q_{1256} & q_{3456} & 0 & q_{5678} \\ q_{1278} & q_{3478} & q_{5678} & 0 \end{bmatrix}. \tag{5.21}$$

*This formula is invariant under the duality map on $\mathrm{Gr}(4,8)$, so it works in both primal and dual Plücker coordinates. The branch locus of $Z$ in $\mathrm{Gr}(2,6)$ is a quartic hypersurface, whose equation is obtained by substituting the $70$ twistor coordinates for the $p_{ijkl}$ in $\mathrm{HL}_\mathcal{V}$.*

**Example 5.4.4.** Consider the map $\mathrm{Gr}(2,8) \to \mathrm{Gr}(2,6)$ given by the totally positive matrix

$$Z \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & -6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The image of the positroid cell $\mathcal{V}_{\geq 0}$ under $Z$ is an interesting Grassmann polytope. The quartic branch locus described above contributes a piece to the boundary of this object. Amplituhedron experts are interested in the quartic equation defining that branch locus.

Let $Y$ be a $6 \times 2$ matrix of unknowns, and write $y_{ij}$ for the $2 \times 2$ minors of $Y$, where $1 \le i < j \le 6$. The substitution (5.13) from primal Plücker coordinates to twistor coordinates is

$$p_{1234} = y_{56}, \ p_{1235} = -y_{46}, \ p_{1236} = y_{45}, \ p_{1237} = y_{45} + y_{46} + y_{56}, \ p_{1238} = y_{45} + 2y_{46} + 3y_{56}, \ \ldots,$$
$$p_{4678} = 2y_{12} + 3y_{13} - y_{15} + 4y_{23} - 2y_{25} - y_{35}, \ \ p_{5678} = y_{12} + 2y_{13} + y_{14} + 3y_{23} + 2y_{24} + y_{34}.$$

After this substitution, $\mathrm{HL}_{\mathcal{V}}$ is a quartic in dual Plücker coordinates $y_{12}, y_{13}, \ldots, y_{56}$ on $\mathrm{Gr}(2,6)$. Its unique expansion in terms of standard monomials has 126 terms. It looks like

$$10y_{12}^2 y_{34} y_{56} - 4y_{12}^2 y_{35}^2 - 4y_{12}^2 y_{35} y_{36} - 12y_{12}^2 y_{35} y_{45} - 14y_{12}^2 y_{35} y_{46} - y_{12}^2 y_{36}^2 - 4y_{12}^2 y_{36} y_{46} - 9y_{12}^2 y_{45}^2$$
$$- 12y_{12}^2 y_{45} y_{46} - 4y_{12}^2 y_{46}^2 + 16y_{12} y_{13} y_{34} y_{56} - 16y_{12} y_{13} y_{35} y_{45} - 20y_{12} y_{13} y_{35} y_{46} + \ \cdots\cdots .$$

Recall that a monomial $y_{i_1 j_1} y_{i_2 j_2} y_{i_3 j_3} y_{i_4 j_4}$ is *standard* if $i_1 \le i_2 \le i_3 \le i_4$ and $j_1 \le j_2 \le j_3 \le j_4$. This quartic equation with 126 standard monomials defines the branch locus in $\mathrm{Gr}(2,6)$.  ◇

*Proof of Proposition 5.4.3.* One checks that the expression on the right hand side of (5.21) is invariant under the duality map given by $p_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \mapsto \mathrm{sign}(\sigma)\, p_{\sigma_5 \sigma_6 \sigma_7 \sigma_8}$ for any permutation $\sigma$ of $\{1, 2, \ldots, 8\}$. We may therefore prove (5.21) for the dual Plücker coordinates on $\mathrm{Gr}(4,8)$.

Any element $P$ of $\mathrm{Gr}(4,8)$ can be represented as the row span of a rank 4 matrix

$$X \ = \ \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} & x_{47} & x_{48} \end{bmatrix}.$$

The Plücker coordinates $p_{ijkl}$ of $P$ are the 70 maximal minors of $X$. For a general subspace $P$ of dimension 4 in $\mathbb{C}^8$, consider all 2-dimensional subspaces $Q \subset P$ such that $Q$ is in $\mathcal{V}$. We claim that there are precisely two such subspaces $Q$. In symbols, $\#(\mathcal{V} \cap \mathrm{Gr}(2, P)) = 2$. We are interested in $P$ such that this number is one.

The argument is analogous to Example 5.3.7, but now for transversal lines to four lines. Let $x_j$ denote the $j$th column of $X$, viewed as a point in $\mathbb{P}^3$. We consider $2 \times 4$ matrices $T$ such that the $2 \times 8$ matrix $TX$ satisfies $q_{12} = q_{34} = q_{56} = q_{78} = 0$. The rows of $T$ span a line in $\mathbb{P}^3$ which intersects the four lines $\overline{x_1 x_2}$, $\overline{x_3 x_4}$, $\overline{x_5 x_6}$ and $\overline{x_7 x_8}$ in $\mathbb{P}^3$. Then $X$ is in the Hurwitz-Lam locus whenever only one such line $T$ exists. Now consider the $6 \times 4$ matrices

$$\begin{aligned} \tilde{X} \ &= \ \begin{bmatrix} x_1 \wedge x_2 & x_3 \wedge x_4 & x_5 \wedge x_6 & x_7 \wedge x_8 \end{bmatrix} \\ (\tilde{X})^* \ &= \ \begin{bmatrix} (x_1 \wedge x_2)^* & (x_3 \wedge x_4)^* & (x_5 \wedge x_6)^* & (x_7 \wedge x_8)^* \end{bmatrix}, \end{aligned}$$

where we order the six rows of each matrix using the ordering $[a_{12} : a_{13} : a_{23} : a_{14} : a_{24} : a_{23}]$ of Plücker coordinates of $x_i \wedge x_{i+1}$. The left kernel of $\tilde{X}$ is represented by a $2 \times 6$ matrix, and gives a line $L$ in $\mathbb{P}^5$. The lines $\overline{x_1 x_2}, \ldots, \overline{x_7 x_8}$ have a single common transversal if and only if $L$ is tangent to $\mathrm{Gr}(2,4)$ in its Plücker embedding; in other words, if $L$ is in the Hurwitz locus of $\mathrm{Gr}(2,4)$. We compute the Hurwitz form of $\mathrm{Gr}(2,4)$ to be

$$\sum_{1 \le i < j \le 6} (-1)^{\sigma(i,j)} L_{i,j} L_{7-j, 7-i}, \tag{5.22}$$

where $L_{i,j}$ are the dual Plücker coordinates of $L$ and $\sigma(i,j)$ is the number of occurrences of 2 or 5 in $(i,j)$. Note that $L_{i,j}$ are actually the primal Plücker coordinates $p_{i,j}(\tilde{X}^T)$ for the row span of the matrix $\tilde{X}^T$ in $\mathrm{Gr}(4,6)$. By the Cauchy-Binet formula, we see

$$\det((\tilde{X})^T \cdot (\tilde{X})^*) \;=\; \sum_{1 \leq i < j \leq 6} p_{i,j}(\tilde{X}^T) p_{i,j}((\tilde{X}^*)^T),$$

which is equal to the expression in (5.22). Finally, by Laplace expansion, the $4 \times 4$ matrix $(\tilde{X})^T \cdot (\tilde{X})^*$ is precisely the matrix on the right side of (5.21). $\qquad\square$

**Remark 5.4.5.** Proposition 5.4.3 extends to all positroid varieties $\mathcal{V}_\beta$ where $\beta = (2,2,\ldots,2)$ is a partition of $n = 2t$ with $t = 2s - 4$ even. The Hurwitz-Lam form $\mathrm{H}_{\mathcal{V}_\beta}$ is obtained from the Hurwitz form of $\mathrm{Gr}(2,s)$ by the same substitution as that for Chow forms in Theorem 5.3.6. Extending to the setting of Remark 5.3.9, we can consider the Hurwitz form of any Grassmannian $\mathrm{Gr}(k,s)$. This becomes the Hurwitz-Lam form for a postroid whose ground set is the disjoint union of nonbases.

We now shift gears and briefly return to the setting of Section 5.1. For any subvariety $\mathcal{V}$ of $\mathbb{P}^{n-1}$ of dimension $d$ and any integer $p \leq d+1$, one associates a subvariety in a Grassmannian:

$$\mathcal{C}_p(\mathcal{V}) \;=\; \big\{\, L \in \mathrm{Gr}(n-p,n) \,:\, \text{the intersection } \mathcal{V} \cap L \text{ is not transverse} \,\big\}. \qquad (5.23)$$

For $p = d + 1$ this is the Chow locus, for $p = d$ it is the Hurwitz locus, and for $p = 1$ it is the discriminant. These varieties $\mathcal{C}_p(\mathcal{V})$ are usually hypersurfaces, and they are known as *coisotropic hypersurfaces* or *higher Chow forms* of $\mathcal{V}$. They were studied by Gel'fand, Kapranov and Zelevinsky in [GKZ94, Chapter 4] and the theory was further developed by Kohn in [Koh21].

It is natural to generalize the definition of higher Chow forms to the setting of Grassmannians. Let $\mathcal{V}$ be a subvariety of $\mathrm{Gr}(k,n)$. For any sufficiently small integer $p$, we consider

$$\mathcal{CL}_{\mathcal{V}}^{(p)} \;=\; \big\{\, L \in \mathrm{Gr}(n-p,n) \,:\, \text{the intersection } \mathcal{V} \cap \mathrm{Gr}(k,L) \text{ is not transverse} \,\big\}. \qquad (5.24)$$

These varieties are called *higher Chow–Lam loci*. Of particular interest are the cases when $\mathcal{CL}_{\mathcal{V}}^{(p)}$ is a hypersurface, defined by a single equation $\mathrm{CL}_{\mathcal{V}}^{(p)}$. We call these the *higher Chow–Lam forms* of $\mathcal{V}$. The case when $p = r - k$ and $\dim(\mathcal{V}) = kp + 1$ recovers the Chow–Lam form; see (5.11). The Hurwitz-Lam form arises for $p = r - k$ and $\dim(\mathcal{V}) = kp$, see (5.20).

**Example 5.4.6** ($p = 2$)**.** Let $\mathcal{V}_\beta = V(q_{12}, q_{34}, q_{56}) \subset \mathrm{Gr}(2,6)$ be the codimension 3 positroid variety given by $\beta = (2,2,2)$. Its Chow–Lam form was derived in Example 5.3.1. We now compute the higher Chow–Lam form $\mathrm{CL}_{\mathcal{V}_\beta}^{(2)}$. The locus $\mathcal{CL}_{\mathcal{V}_\beta}^{(2)}$ consists of all subspaces $L \in \mathrm{Gr}(4,6)$ such that the curve $\mathcal{V}_\beta \cap \mathrm{Gr}(2,L)$ is singular. This corresponds to cutting the Grassmannian $\mathrm{Gr}(2,4)$ with three hyperplanes in $\mathbb{P}^5$ such that the intersection is a singular curve. The corresponding equation $\mathrm{C}_{\mathrm{Gr}(2,4)}^{(3)}$ is irreducible of degree 6 in primal Stiefel coordinates. If we replace these by the columns of the $6 \times 3$ matrix $\tilde{X} = \big[ x_1 \wedge x_2 \;\; x_3 \wedge x_4 \;\; x_5 \wedge x_6 \big]$,

then we obtain a reducible polynomial of degree 12 in the entries of the $4 \times 6$ matrix $X = [x_1, x_2, \ldots, x_6]$:

$$\mathrm{CL}^{(2)}_{\mathcal{V}_\beta} = \mathrm{C}^{(3)}_{\mathrm{Gr}(2,4)} = p_{12}\,p_{34}\,p_{56} = \det(x_3, x_4, x_5, x_6)\,\det(x_1, x_2, x_5, x_6)\,\det(x_1, x_2, x_3, x_4).$$

This is similar to the identity in (5.18). The factorization has a geometric meaning: Consider the curve of lines that intersect three given lines in $\mathbb{P}^3$. This curve is smooth if and only if the three lines are pairwise disjoint. Thus $\mathcal{CL}^{(2)}_{\mathcal{V}_\beta}$ has three irreducible components. $\diamond$

In this chapter we introduced a new discriminant, the Chow–Lam form. This was inspired by computations in particle physics, but is of intrinsic interest in computational algebraic geometry. In the following chapters we ask purely algebra-geometric questions about it, inspired by properties of the Chow form.

# Chapter 6

# Chow–Lam Recovery

This chapter is based on the paper [PR25] with Kristian Ranestad, and is almost identical, except that some preliminary material has been removed.

We are interested in projections between Grassmannians induced by projections of the underlying projective spaces. That is, fix an $n \times r$ matrix $Z$ of rank $r$ and consider the map

$$
\begin{aligned}
Z : \mathrm{Gr}(k, n) &\dashrightarrow \mathrm{Gr}(k, r) \\
[M] &\mapsto [MZ]
\end{aligned}
\tag{6.1}
$$

where $M$ is a $k \times n$ matrix and $[M]$ is its rowspan. When $k = 1$ we recover linear projections between projective spaces. These are exactly the linear projections which arise in the context of the amplituhedron; see Definition 1.1.1.

In this paper, $\mathcal{V}$ will be a subvariety of $\mathrm{Gr}(k, n)$ of dimension $k(r - k) - 1$ for some $r \leq n$. Let $Z(\mathcal{V})$ denote the closure of the image of $\mathcal{V}$ under the projection map induced by $Z$. Then for a general $Z$ the variety $Z(\mathcal{V})$ is expected to be a hypersurface in the target Grassmannian, and is thus cut out by a single equation.

We ask the question: can $\mathcal{V}$ be recovered from the data of *all* possible projections $Z(\mathcal{V})$? Let $Z^{-1}(Z(\mathcal{V}))$ denote the pre-image of $Z(\mathcal{V})$ in $\mathrm{Gr}(k, n)$. We have the containment

$$
\mathcal{V} \subseteq \bigcap_Z Z^{-1}(Z(\mathcal{V})).
\tag{6.2}
$$

If $k = 1$, equality holds in (6.2), for reasons we explore in the coming paragraphs. In general the right side may be strictly larger.

We now explain how this question is related to the recovery of a variety from its Chow–Lam form. If $A, B$ are subvarieties of $\mathbb{P}^n$, we define their *join* $A \vee B$ to be the closure of the union of all lines $\overline{ab}$ spanned by distinct points $a$ in $A$ and $b$ in $B$. The fibers of the projections $Z$ of the form (6.1) can be described using join. Let $K_Z \in \mathrm{Gr}(n - r, n)$ be the projectivized kernel of the $r \times n$ matrix $Z^T$.

**Lemma 6.0.1.** *Fix a linear space $P \in \mathrm{Gr}(k, n)$, and assume that $K_Z$ does not intersect $P$. Then $P \vee K_Z$ is in $\mathcal{CL}_{\mathcal{V}}$ if and only if $P$ is in $Z^{-1}(Z(\mathcal{V}))$.*

*Proof.* Observe that $Z^{-1}(Z(P))$, as a subset of $\mathrm{Gr}(k, n)$, is exactly the Grassmannian $\mathrm{Gr}(k, \widehat{P \vee K_Z})$. So $Z(P)$ is in $Z(\mathcal{V})$ only if $P \vee K_Z$ contains as a subspace some $Q \in \mathcal{V}$. $\quad\square$

Inspired by Lemma 6.0.1, we define the *recovered variety* of a variety in $\mathrm{Gr}(k, n)$.

**Definition 6.0.2** (Recovered variety)**.** Let $\mathcal{V} \subset \mathrm{Gr}(k, n)$ be a variety of dimension $k(r-k)-1$ for some $r \leq n$. We define the algebraic set

$$W_\mathcal{V} := \{P \in \mathrm{Gr}(k, n) \ : \ Q \in \mathcal{CL}_\mathcal{V} \text{ whenever } Q \supset P\} \tag{6.3}$$

and call it the *recovered variety* of $\mathcal{V}$.

For simplicity we will call $W_\mathcal{V}$ a variety, but in general it may be reducible.

**Lemma 6.0.3.** *The varieties $W_\mathcal{V}$ and $\bigcap_Z Z^{-1}(Z(\mathcal{V}))$ are equal.*

*Proof.* We recall from Section 5.2 that the dimension of $Q$ is $n - r + k - 1$. Note that if $Q \supset P$, then $Q \supset P \vee A$ for every $A \in \mathrm{Gr}(n - r, \hat{Q})$, and each such $A$ is the projectivized kernel $K_Z$ for some $n \times r$ matrix $Z$. We conclude using Lemma 6.0.1. $\quad\square$

For the Chow form, we have that $W_\mathcal{V} = \mathcal{V}$. That is, any projective variety can be uniquely recovered from its Chow form. In fact, this was the original motivation for Chow and van der Waerden to study these polynomials [CW37]. However, this is not true for the Chow–Lam form: two different varieties may have the same Chow–Lam form, as in Example 6.0.4.

**Example 6.0.4** (A ruled quadric)**.** Let $X$ be a quadric surface in $\mathbb{P}^3$. Then $X$ has two rulings, which give two curves $\mathcal{V}$ and $\mathcal{V}'$ in $\mathrm{Gr}(2, 4)$. Recall from Example 1.3.3 that the Chow–Lam locus of each of these curves is the dual variety $X^\vee$ to $X$. Thus $W_\mathcal{V}$ contains $\mathcal{V} \cup \mathcal{V}'$. In particular, unique recovery from (6.1) fails in this example. $\quad\diamond$

The structure of this chapter is as follows. In Section 6.1 we will review the theory of Chow–Lam forms and their computation. In Section 6.2 we will discuss a general criterion for a point in $\mathrm{Gr}(k, n)$ to lie in $W_\mathcal{V}$. The remaining three sections are devoted to examples. Section 6.3 discusses curves and surfaces, Section 6.4 describes families of varieties with the same Chow–Lam form, and Section 6.5 describes $W_\mathcal{V}$ when $\mathcal{V}$ is a Schubert hyperplane section of the Grassmannian.

# 6.1 Computation of recovered varieties

Suppose that $\mathcal{V} \subset \mathrm{Gr}(k, n)$ is a variety of dimension $k(r - k) - 1$ for some $r \geq k$. Then the Chow–Lam form of $\mathcal{V}$ can be obtained by first computing equations of the incidence variety

$$\Phi = \{(P, Q) : P \subset Q\} \subset \mathrm{Gr}(k, n) \times \mathrm{Gr}(n - r + k, n).$$

We then intersect $\Phi$ with $\mathcal{V} \times \mathrm{Gr}(n - r + k, n)$ and project to the latter factor. Algebraically, this corresponds to taking the sum of the ideal of $\Phi$ with the ideal of $\mathcal{V}$, saturating appropriately, and eliminating the variables in $\mathrm{Gr}(k, n)$.

**Example 6.1.1** (Equations of incidence variety). Suppose $k = 2, n = 4$, and $r = 3$. The incidence variety $\Phi$ will consist of pairs of lines and planes in $\mathbb{P}^3$ such that the plane contains the line. If $p_i$ are the primal coordinates of the plane and $q_{ij}$ are the dual coordinates of the line, then the ideal of $\Phi$ is generated by

$$
\begin{aligned}
&q_{12}p_1 - q_{23}p_3 - q_{24}p_4, \quad q_{12}p_2 + q_{13}p_3 + q_{14}p_4, \\
&q_{13}p_1 + q_{23}p_2 - q_{34}p_4, \quad q_{14}p_1 + q_{24}p_2 + q_{34}p_3.
\end{aligned}
\tag{6.4}
$$

$\diamond$

Given the Chow–Lam form $\mathrm{CL}_{\mathcal{V}}$ of a variety $\mathcal{V}$, the variety $W_{\mathcal{V}}$ can be computed as follows. We expand the dual Plücker coordinates of $A \vee P \in \mathrm{Gr}(n - r + k, n)$ in terms of dual Plücker coordinates of $A \in \mathrm{Gr}(n - r, n)$ and $P \in \mathrm{Gr}(k, n)$, then collect coefficients of monomials in the $q_I(A)$. When $A \vee P \in \mathcal{CL}_{\mathcal{V}}$, these coefficients are polynomials in the variables $q_I(P)$ which define the variety $W_{\mathcal{V}}$. Note that this process produces not only a variety, but a scheme; indeed, we define the *recovered scheme* $\mathcal{W}_{\mathcal{V}}$ to be the scheme defined by these polynomials. In the classical case of the Chow form, cf. [Cat92, Theorem 1.14], the scheme $\mathcal{W}_{\mathcal{V}}$ is equal to $\mathcal{V}$ exactly at the smooth points of $\mathcal{V}$ and has embedded components at the singular points. We can see this in Example 6.1.2. In the Chow–Lam case the situation is more complicated: even for a smooth subvariety $\mathcal{V}$ of the Grassmannian with $W_{\mathcal{V}} = \mathcal{V}$, there may be embedded components in $\mathcal{W}_{\mathcal{V}}$, as in Example 6.1.3.

**Example 6.1.2** (Singular quintic in $\mathbb{P}^3$). Consider the curve $\mathcal{V} \subset \mathbb{P}^3$ given parametrically by $[s : t] \mapsto [s^5 : s^4t : s^3t^2 : t^5]$. Let $\mathbb{P}^3$ have coordinates $x_1, x_2, x_3, x_4$. Then $\mathcal{V}$ has a singularity of Jacobian degree 4 at $[0 : 0 : 0 : 1]$ and is otherwise smooth; that is, the vanishing of the Jacobian and the equation for $\mathcal{V}$ define an ideal of length 4 whose variety is that point. Because the curve is given to us parametrically, we can also compute the Chow form parametrically: it is the closure of rowspans of matrices

$$
L_s = \begin{bmatrix} 1 & t & t^2 & t^5 \\ a & b & c & d \end{bmatrix}
$$

and obtained by eliminating $t, a, b, c, d$. In primal Plücker coordinates this gives

$$
\begin{aligned}
\mathrm{C}_{\mathcal{V}} = {}& p_{14}^5 - p_{13}^3 p_{14} p_{24} + 3 p_{12} p_{13} p_{14}^2 p_{24} + p_{12}^3 p_{24}^2 \\
& + p_{13}^4 p_{34} - p_{12}^3 p_{23} p_{34} - 4 p_{12} p_{13}^2 p_{14} p_{34} + 2 p_{12}^2 p_{14}^2 p_{34}.
\end{aligned}
$$

We view the $p_{ij}$ as primal Plücker coordinates of the line $x \vee y$ where $x$ and $y$ are points in $\mathbb{P}^3$. Then $x$ is in $\mathcal{V}$ if and only if $x \vee y$ is in $\mathcal{C}_{\mathcal{V}}$ for all $y \in \mathbb{P}^3$. Thus we can make the change of coordinates $p_{12} = x_3 y_4 - x_4 y_3$, etc. Collecting coefficients of monomials in the $y$ variables gives us an ideal which is the intersection of two primary components. One is the ideal of $\mathcal{V}$, and the other is the degree 16 ideal

$$
(x_4^2, x_3^3 x_4, x_2 x_3^2 x_4, x_2^2 x_3 x_4, x_2^3 x_4, x_3^4, x_2 x_3^3, x_2^2 x_3^2, x_2^3 x_3 - x_1 x_3^2 x_4, x_2^4 - x_1 x_3^3)
$$

whose variety is the point $[0 : 0 : 0 : 1]$.

$\diamond$

**Example 6.1.3** (Curve in $\mathrm{Gr}(2,4)$)**.** Consider the curve $\mathcal{V}$ in $\mathrm{Gr}(2,4)$ given by tangent lines to the quintic in Example 6.1.2. In dual Plücker coordinates, this curve is given by

$$\mathcal{V} = V(q_{24}^2 - 4q_{23}q_{34}, 3q_{14}q_{24} - 10q_{13}q_{34}, q_{13}q_{24} - 6q_{12}q_{34}, 8q_{14}^2 - 25q_{23}q_{24},$$
$$q_{23}q_{14} - 5q_{12}q_{34}, 15q_{23}^2 - 4q_{13}q_{14}, 2q_{13}q_{23} - 3q_{12}q_{24}, 5q_{13}^2 - 9q_{12}q_{14}).$$

Then $\mathcal{V}$ has a singular locus consisting of two points $[0:0:0:0:0:1]$ and $[1:0:0:0:0:0]$, each of Jacobian degree 2. The Chow–Lam form of $\mathcal{V}$ in $\mathrm{Gr}(3,4) = \mathbb{P}^3$ is the degree 7 form given in primal Plücker coordinates by

$$\mathrm{CL}_{\mathcal{V}} = 16p_2^3 p_3^4 + 108p_1^2 p_3^5 - 128p_2^4 p_3^2 p_4$$
$$- 900p_1^2 p_2 p_3^3 p_4 + 256p_2^5 p_4^2 + 2000p_1^2 p_2^2 p_3 p_4^2 + 3125p_1^4 p_4^3.$$

Here $\mathrm{CL}_{\mathcal{V}}$ can be computed by using the equations in Example 6.1.1 and eliminating. The ideal of the recovered scheme has three primary components of degrees $14, 42$, and $52$. Their corresponding varieties are the curve $\mathcal{V}$ and the two singular points. This is an example where $W_{\mathcal{V}}$ and $\mathcal{V}$ are the same as algebraic sets, but scheme-theoretically different even at the smooth points, as seen by the degree. $\diamond$

To keep our discussion simple, we will restrict ourselves to considering the recovered variety $W_{\mathcal{V}}$ as an algebraic set. We end this section by noting that the dimension of $W_{\mathcal{V}}$ may be higher than that of $\mathcal{V}$.

**Example 6.1.4** (High dimension $W_{\mathcal{V}}$)**.** Choose a basis $e_1, ..., e_n$ of $\mathbb{C}^n$. Let $\mathcal{W} \subset \mathrm{Gr}(k,n)$ be the Schubert variety of subspaces meeting $e_n$, which we now view as a point in $\mathbb{P}^{n-1}$. Then $\mathcal{W}$ is isomorphic to the Grassmannian $\mathrm{Gr}(k-1, n-1)$. Let $\mathcal{V}$ be a generic $(k-1)$-dimensional subvariety of $\mathcal{W}$. We claim that $\mathcal{W}$ and $\mathcal{V}$ have the same $Z$-projections for a generic matrix $Z$ of dimension $n \times (k+1)$. Thus $\mathcal{W}$ will appear as a component in $W_{\mathcal{V}}$.

To see this, let $Z_n$ be the last row of $Z$. Then we may view $Z_n$ as a point in $\mathbb{P}^k$. The projection $Z(\mathcal{W})$ consists of all $(k-1)$-spaces in $\mathbb{P}^k$ meeting the point $Z_n$. So $Z(\mathcal{W})$ is a hyperplane in $\mathrm{Gr}(k, k+1) = (\mathbb{P}^k)^\vee$. To conclude it suffices to show that $Z(\mathcal{V})$ has the same dimension as $\mathcal{V}$ for generic $\mathcal{V} \subset \mathcal{W}$. But this follows by [Lam16a, Proposition 4.8], which gives cohomological conditions on $\mathcal{V}$ for the dimension of the projection to agree with the dimension of the variety.

How much bigger is $\mathcal{W}$? It has codimension $n-k$ and dimension $(k-1)(n-k)$ in $\mathrm{Gr}(k,n)$. Recall that $\mathcal{V}$ has dimension $k+1$. Thus as $k$ increases, we get a family of examples which show that $W_{\mathcal{V}}$ can be arbitrarily large compared to $\mathcal{V}$. $\diamond$

## 6.2 A recovery criterion

Fix a variety $\mathcal{V} \subset \mathrm{Gr}(k,n)$ of dimension $k(r-k) - 1$ for some $r$. In this section, we will establish a general criterion for a point $P \in \mathrm{Gr}(k,n)$ to be in $W_{\mathcal{V}}$.

Fix a linear space $P \subset \mathbb{P}^{n-1}$ of dimension $k-1$. Then we may stratify $\mathcal{V}$ as follows. Let

$$\mathcal{V}_{P,i} = \{P' \in \mathcal{V} : \dim P' \cap P \geq i\} \subset \mathrm{Gr}(k, n)$$

be the Schubert variety of spaces in $\mathcal{V}$ which meet $P$ in dimension at least $i$. We take the empty set to have dimension $-1$. Then we have

$$\mathcal{V} = \mathcal{V}_{P,-1} \supset \mathcal{V}_{P,0} \supset ... \supset \mathcal{V}_{P,k-1},$$

where the last term is nonempty exactly when $P$ itself is in $\mathcal{V}$. We begin with a lemma on the dimension of certain Schubert varieties, then state our main theorem. Let $Q \subset \mathbb{P}^{n-1}$ be a subspace of dimension $m$ and assume $l > m$. Then we denote by $\Omega(Q) \subset \mathrm{Gr}(l, n)$ the Schubert variety of subspaces that contain $Q$.

**Lemma 6.2.1** (Dimension). *Suppose $Q \subset \mathbb{P}^{n-1}$ is a subspace of dimension $m$. Then the variety $\Omega(Q)$ has dimension $(l - m - 1)(n - l)$ in $\mathrm{Gr}(l, n)$.*

*Proof.* The Schubert variety $\Omega(Q)$ is isomorphic to the Grassmannian $\mathrm{Gr}(l-m-1, n-m-1)$, so has dimension $(l - m - 1)(n - l)$ in $\mathrm{Gr}(l, n)$. $\qquad\qquad\square$

**Theorem 6.2.2** (Recovery criterion). *Let $\mathcal{V} \subset \mathrm{Gr}(k, n)$ be a subvariety of dimension $k(r - k) - 1$. If a linear space $P \in \mathrm{Gr}(k, n)$ is in the recovery $W_{\mathcal{V}}$, then for some $0 \leq i \leq k - 1$, the variety $\mathcal{V}_{P,i}$ has dimension at least $(k - i - 1)(r - k)$.*

*Proof.* The proof is essentially a dimension argument. Consider the Schubert variety $\Omega(P) \subset \mathrm{Gr}(n - r + k, n)$ of $(n - r + k - 1)$-spaces containing $P$. Observe that a linear space $P$ is in the recovery if and only if every point of $\Omega(P)$ is also a point of $\Omega(P')$ for some $P' \in \mathcal{V}$.

So we consider the incidence correspondence

$$I_P = \{(R, P') \mid R \in \Omega(P) \cap \Omega(P'), P' \in \mathcal{V}\} \subset \mathrm{Gr}(n - r + k, n) \times \mathrm{Gr}(k, n),$$

with projections $\pi : I_P \to \Omega(P)$ and $\pi' : I_P \to \mathcal{V}$. Then $P$ is in the recovery if and only if $\pi$ is surjective. The latter is the case only if the dimension of $I_P$ is at least $(n - r)(r - k) = \dim \Omega(P)$. Now $I_P$ may have several components since the fibers of $\pi'$ may differ in dimensions. The fiber $(\pi')^{-1}(P')$ is the variety $\Omega(P \wedge P') \subset \mathrm{Gr}(n - r + k, n)$. The dimension of this fiber depends on the dimension of the intersection $P \cap P'$. Observe that

$$\dim P' \vee P = \dim P' + \dim P - \dim(P' \cap P).$$

When $P' \in \mathcal{V}_{P,i} \backslash \mathcal{V}_{P,i+1}$, the span $P \wedge P'$ of $P$ and $P'$ has dimension $2k - i - 2$, so $\Omega(P \cap P')$ has dimension $(n - r - k + i + 1)(r - k)$. Therefore the preimage over $\mathcal{V}_{P,i}$ dominates $\Omega(P)$ only if $\dim \mathcal{V}_{P,i} + (n - r - k + i + 1)(r - k) \geq (n - r)(r - k)$, i.e. when $\dim \mathcal{V}_{P,i} \geq (k - i - 1)(r - k)$. $\quad\square$

For example, let $\mathcal{V}$ be a surface in $\mathrm{Gr}(3, n)$, whose points represent planes in $\mathbb{P}^{n-1}$. Then $\dim \mathcal{V} = 2 = 3(r - 3) - 1$, so $r = 4$. If a plane $P$ is in $W_{\mathcal{V}}$, then by Theorem 6.2.2 at least one of the following must be true:

- $(i = 0)$ : $P$ meets each $P' \in \mathcal{V}$ in a point, or

- $(i = 1)$ : $P$ meets a one-dimensional family of $P' \in \mathcal{V}$ in a line, or

- $(i = 2)$ : $P$ is in $\mathcal{V}$, i.e. meets a point in $\mathcal{V}$ in a plane.

The dimension condition in Theorem 6.2.2 may be satisfied for multiple $i$. For example, suppose all planes $P'$ in $\mathcal{V}$ contain a fixed line $L$ in $\mathbb{P}^{n-1}$. Then for any plane $P$ not in $\mathcal{V}$ containing $L$, the dimension condition is satisfied for both $i = 0$ and $i = 1$.

Theorem 6.2.2 gives a necessary condition for $P$ to be in the recovery, but it is not sufficient in general. Even if the variety $\mathcal{V}_{P,i}$ has high dimension, the map $\pi$ may have high-dimensional fibers and so may not be surjective.

**Example 6.2.3** (Counterexample to the converse of Theorem 6.2.2)**.** This example will show that the condition in Theorem 6.2.2 is not sufficient for $P$ to be in $W_{\mathcal{V}}$. Let $n$ be large, and let $e_i$ be the $i$th standard basis vector. Fix the plane $Q := e_1 \vee e_2 \vee e_3$ and the 3-space $P := e_1 \vee e_2 \vee e_3 \vee e_4$. Next, we let $\mathcal{V}_1$ be a 4-dimensional subvariety of $\mathrm{Gr}(4, n)$ with the property that all $P'$ in $\mathcal{V}_1$

  i) meet $Q$ in at least a line

  ii) meet $e_{n-1} \vee e_n$ in at least a point.

Finally, choose a general 7-dimensional variety $\mathcal{V} \subset \mathrm{Gr}(4, n)$ containing $\mathcal{V}_1$. Then $r = 6$ and the Chow–Lam locus $\mathcal{CL}_{\mathcal{V}}$ lives in $\mathrm{Gr}(n - 2, n)$. The dimension of $\mathcal{V}_{P,1}$ is at least $(k - 1 - 1)(r - k) = 4$, which per Theorem 6.2.2 is necessary for $P$ to be in the recovery; indeed, the dimension of $\mathcal{V}_{P,0}$ is lower than 6 if $\mathcal{V}$ is chosen generically. However, we will see that $P$ is not in the recovery.

To see this, take the codimension two subspace $R \subset \mathbb{P}^{n-1}$ given by the span of all vectors except $e_{n-1}$ and $e_n$. Then $R$ contains $P$, but by construction cannot contain any $P'$ in $\mathcal{V}_1$. If $\mathcal{V}$ is chosen generically to contain $\mathcal{V}_1$, then $R$ will not contain any $P'$ in $\mathcal{V}$ either. Thus $\pi : I_P \to \Omega(P)$ is not surjective, and $P$ is not in the reecovery $W_{\mathcal{V}}$. $\diamond$

## 6.3 Chow–Lam locus as a dual variety

Given a variety $\mathcal{V} \subset \mathrm{Gr}(k, n)$, we define $X_{\mathcal{V}} \subset \mathbb{P}^{n-1}$ to be the union of all $(k - 1)$-spaces in $\mathcal{V}$, or equivalently the image under the projection to $\mathbb{P}^{n-1}$ of the incidence

$$\{(x, P) \mid x \in P \subset \mathbb{P}^{n-1}, P \in \mathcal{V}\} \ \subset \ \mathbb{P}^{n-1} \times \mathrm{Gr}(k, n).$$

The following proposition is a generalization of Example 6.0.4 of the quadric surface.

**Proposition 6.3.1.** *Let $\mathcal{V}$ be a subvariety of $\mathrm{Gr}(k, n)$ of dimension $k-1$. Then the dual $X_{\mathcal{V}}^{\vee}$ is contained in $\mathcal{CL}_{\mathcal{V}}$, with equality whenever their dimensions agree. In this case a $(k-1)$-space $P$ belongs to $W_{\mathcal{V}}$ if and only if its dual linear space $P^{\perp}$ is contained in $X_{\mathcal{V}}^{\vee}$.*

*Proof.* Note that if $n$ is less than $2k$, then $X_\mathcal{V} = \mathbb{P}^{n-1}$ and its dual is empty, meaning the proposition has no content. For the rest of the proof we suppose that $n \geq 2k$. Here $r = k+1$, so the Chow–Lam locus is a hypersurface in $\mathrm{Gr}(n-1, n) \cong (\mathbb{P}^{n-1})^\vee$. Take any tangent space to $X_\mathcal{V}$ at a smooth point $x$. Then it must contain all linear spaces in $X_\mathcal{V}$ through $x$. In particular it contains all $(k-1)$-dimensional linear spaces through $x$ that belong to $\mathcal{V}$. But by definition, $x$ is in at least one $(k-1)$-space in $\mathcal{V}$. Thus any hyperplane that contains such a tangent space, i.e. any hyperplane that belong to $X_\mathcal{V}^\vee$, is in the Chow–Lam locus. Both $X_\mathcal{V}^\vee$ and $\mathcal{CL}_\mathcal{V}$ are irreducible, so we have equality whenever their dimensions coincide. In this case the last statement of the proposition is immediate. □

Let us apply Theorem 6.2.2 to curves. When a curve $\mathcal{V} \subset \mathrm{Gr}(k, n)$ has a Chow–Lam form, then $k(r-k) - 1 = 1$. Thus either $k = 1$ and we are in the ordinary case of Chow forms, or $k = 2$ and $r = 3$ and the Chow–Lam form defines a hypersurface in $\mathrm{Gr}(n-1, n)$. In the latter case, the Proposition 6.3.1 applies, and the Chow–Lam form defines the dual variety $X_\mathcal{V}^\vee$ whenever the latter is non-degenerate.

More generally, when $\mathcal{V} \subset \mathrm{Gr}(k, n)$ has a Chow–Lam form and $k(r-k) - 1 = p - 1$, where $p$ is prime, then $k = 1$ and we are in the case of Chow forms. Else, $k = p, r = p + 1$ and the Chow–Lam locus lives in $\mathrm{Gr}(n-1, n)$. We have the following corollary of Proposition 6.3.1:

**Corollary 6.3.2.** Fix a variety $\mathcal{V} \subset \mathrm{Gr}(k, n), k > 1$ of dimension $p - 1$ for $p$ prime. Then $\mathcal{V}$ has a Chow–Lam form only if $k = p$ and $\mathcal{CL}_\mathcal{V} \subset (\mathbb{P}^{n-1})^\vee$. In this case, a $(p-1)$-space $P$ is in the recovery $W_\mathcal{V}$ if and only if $P^\perp \subset X_\mathcal{V}^\vee$.

In the case $p = 2$, i.e. when $\mathcal{V}$ is a curve with a Chow–Lam form, then $k = 2$. With this assumption we can prove a stronger statement.

**Corollary 6.3.3.** Let $\mathcal{V} \subset \mathrm{Gr}(2, n)$ be a curve such that the ruled surface $X_\mathcal{V}$ is not a cone. Then a line $L \in \mathbb{P}^{n-1}$ is in the recovery $W_\mathcal{V}$ if and only if $L \subset X_\mathcal{V}$.

*Proof.* When the surface scroll $X_\mathcal{V}$ is not a cone, the dual variety $X_\mathcal{V}^\vee$ is a hypersurface which coincides with the Chow–Lam locus. If $L$ is a line contained in the surface $X_\mathcal{V}$, then every hyperplane that contains $L$ will contain a line that belongs to $\mathcal{V}$, so $L$ will belong to the recovery $W_\mathcal{V}$. For the converse, we will show that if $L$ is not contained in $X_\mathcal{V}$ then we can find a hyperplane that contains $L$ but no line in $\mathcal{V}$.

First note that since $X_\mathcal{V}$ is not a cone, only finitely many lines in $\mathcal{V}$ intersect $L$. Now, consider a general codimension two linear space $Q$ that contains $L$. It will intersect $X_\mathcal{V}$ in finitely many points. So there are only finitely many lines in $\mathcal{V}$ that intersect $Q$. But then there are only finitely many hyperplanes that contain both $Q$ and a line in $\mathcal{V}$. Therefore a general hyperplane containing $Q$ will contain no line in $\mathcal{V}$. □

**Example 6.3.4** (Hirzebruch surface). Consider the Hirzebruch surface

$$F_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)).$$

Since $F_{a-1}$ is the projectivization of a bundle, there is a map $\pi : F_a \to \mathbb{P}^1$ whose fibers are isomorphic to $\mathbb{P}^1$. Then $F_a$ is ruled by these fibers, all of which meet the curve $L \subset F_a$ corresponding to the line subbundle $\mathcal{O}_{\mathbb{P}^1}(1)$. Note that

$$H^0(F_a, \mathcal{O}_{F_a}(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)).$$

The latter space has dimension $a + 4$. Thus $\mathcal{O}_{F_a}(1)$ gives an embedding of $F_a$ into $\mathbb{P}^{a+3}$. Let $\mathcal{V}$ be the curve in $\mathrm{Gr}(2, a+4)$ whose points are the lines in the ruling of $X$. Then the image of $L$ is a line and the recovery of $\mathcal{V}$ will contain this line as a point in $\mathrm{Gr}(2,n) \setminus \mathcal{V}$.

For a concrete example, let $a = 1$. Then $X$ is the closure of the embedding

$$(\mathbb{C}^*)^2 \to \mathbb{P}^4$$
$$(s : t) \mapsto [1 : s : s^2 : st : s^2 t].$$

Each line in the ruling is given by fixing $s$. Parametrically, they are rowspans of

$$L_s = \begin{bmatrix} 1 & s & s^2 & 0 & 0 \\ 1 & s & s^2 & s & s^2 \end{bmatrix}.$$

These lines form a curve $\mathcal{V} \subset \mathrm{Gr}(2,5)$ given by the Plücker relations and

$$V(q_{45}, q_{34} - q_{25}, q_{24} - q_{15}, q_{23}, q_{13}, q_{12}).$$

The Chow–Lam form may be calculated parametrically by adding two rows to $L_s$, and is $p_3 p_4^2 - p_2 p_4 p_5 + p_1 p_5^2$.

The recovered scheme will be the original curve as well as a fat point of length 25, whose radical ideal consists of all Plücker coordinates except for $q_{45}$. This is the line spanned by $[0 : 0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 0 : 1]$ obtained in the closure of the embedding by allowing $t$ to approach infinity. $\diamond$

In a surface scroll $X_\mathcal{V}$, there may even be two lines $L_1$ and $L_2$ in the scroll that are not rulings. In this case every line in the ruling is in the linear span of $L_1$ and $L_2$, i.e. a $\mathbb{P}^3$. Then $n = 4$ and the scroll $X_\mathcal{V}$ is a surface of degree $d_1 + d_2$ in $\mathbb{P}^3$ with multiplicity $d_1$ along $L_1$ and multiplicity $d_2$ along $L_2$. We explain these multiplicities as follows. We have that $\mathcal{V} \subset \mathrm{Gr}(2,4)$ is contained in the quadric surface of lines meeting $L_1$ and $L_2$. The surface $X_\mathcal{V}$ is the birational image of $\mathbb{P}_\mathcal{V}(\mathcal{L}_1 + \mathcal{L}_2)$ in $\mathbb{P}^3$. Here $\mathcal{L}_1$ is a line bundle of degree $d_i$ and $L_i$ is the image of $\mathcal{V}$ by a basepoint free pencil of sections of $\mathcal{L}_i$. The degree of $X_\mathcal{V}$ is then computed on $\mathbb{P}_\mathcal{V}(\mathcal{L}_1 + \mathcal{L}_2)$, or in $\mathbb{P}^3$. The map of $\mathcal{V}$ to $L_i$ has degree $d_i$, so $X_\mathcal{V}$ must have multiplicity $d_i$ along $L_i$. The degree of $X_\mathcal{V}$ in $\mathbb{P}^3$ is computed as a sum of intersection multiplicities with a line. A general line that intersect $L_1$ and $L_2$ will meet $X_\mathcal{V}$ only in these lines, and the sum of intersection multiplicities is the sum of the multiplicities of $X_\mathcal{V}$ along these lines, i.e. $d_1 + d_2$.

**Example 6.3.5** (Maps to a curve and a line)**.** Consider a curve $\mathcal{C}$ with a $d_1 : 1$ map $\varphi_1$ to $\mathbb{P}^1$ and a birational map $\varphi_2$ to $\mathbb{P}^{n-3}$ with image a curve $C$ of degree $d_2$. Next, we embed
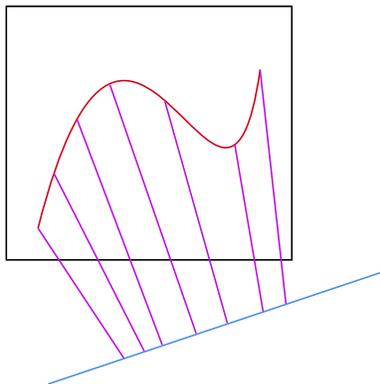
Figure 6.1: Construction of a rational scroll

$\mathbb{P}^{n-3}$ and $\mathbb{P}^1$ as non-intersecting linear spaces in $\mathbb{P}^{n-1}$. Let $X$ be obtained by joining, for each $c \in \mathcal{C}$, the points $\varphi_1(c)$ and $\varphi_2(c)$ in $\mathbb{P}^{n-1}$. Then the two maps $\varphi_1$ and $\varphi_2$ define a map $\mathcal{C} \to \mathrm{Gr}(2, n)$ with image $\mathcal{V}$ such that $X = X_\mathcal{V}$. The recovery of $\mathcal{V}$ contains the line $\varphi_1(C)$ in $X$ that does not belong to $\mathcal{V}$. ◇

Analogously, we find surfaces $\mathcal{V} \subset \mathrm{Gr}(3, n), n \geq 10$ for which $W_\mathcal{V} \neq \mathcal{V}$.

**Example 6.3.6.** (Maps to a surface and a plane). Suppose an abstract (not embedded) surface $\mathcal{S}$ has two very ample linear systems $L_1, L_2$ and a basepoint free 2-dimensional linear system $L_3$. Let $X_\mathcal{V}$ be the 4-dimensional $\mathbb{P}^2$-scroll spanned by the images of $\mathcal{S}$ by the sum of the three linear systems (composed possibly with a general linear projection to $\mathbb{P}^{n-1}$ with $n \geq 10$). This gives us an embedding $\mathcal{S} \to \mathcal{V} \subset \mathrm{Gr}(3, n)$ such that the plane image of $\mathcal{S}$ by $L_3$ belongs to the $W_\mathcal{V}$ but not to $\mathcal{V}$. ◇

## 6.4 Multi-ruled varieties

In this section we generalize Example 6.0.4 and construct subvarieties of $\mathrm{Gr}(k, n)$ which have the same Chow–Lam form. We begin with an immediate corollary of Proposition 6.3.1.

**Corollary 6.4.1** (Multi-ruled varieties). Let $\mathcal{V}_1, \mathcal{V}_2$ be subvarieties of $\mathrm{Gr}(k, n)$ of dimension $k - 1$. Let $X_i \subset \mathbb{P}^{n-1}$ be the closure of the union of linear spaces in $\mathcal{V}_i$ for $i = 1, 2$. Suppose that the duals $X_i^\vee$ are non-degenerate. Then $X_1 = X_2$ if and only if $\mathcal{V}_1$ and $\mathcal{V}_2$ have the same Chow–Lam locus.

**Example 6.4.2** (Segre varieties). Consider the Segre embedding $\Sigma_K := \mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{k^2-1}$. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the two $(k-1)$-dimensional subvarieties of $\mathrm{Gr}(k, k^2)$ that parameterize

$(k-1)$-spaces in $\Sigma_K$. That is, $\mathcal{V}_1$ parameterizes the $(k-1)$-spaces in the second factor,

$$\mathcal{V}_1 = \left\{ \mathrm{Seg}(\{p\} \times \mathbb{P}^{k-1}) \in \mathrm{Gr}(k, k^2) \ : \ p \in \mathbb{P}^{k-1} \right\},$$

while $\mathcal{V}_2$ parameterizes the $(k-1)$-spaces in the first factor. Then the ruled varieties $X_i$ as in the Corollary 6.4.1 coincide with $\Sigma_K$. In the special case of $k = 2$, the resulting variety $\Sigma_k$ is a quadric surface in $\mathbb{P}^3$, and we recover Example 6.0.4.

More generally, the dual variety of $\Sigma_K$ is non-degenerate [Gel+94, Theorem 0.1]. Thus Corollary 6.4.1 tells us that $\mathcal{V}_1$ and $\mathcal{V}_2$ both have Chow–Lam locus equal to $\Sigma_k^\vee$.  ◇

**Example 6.4.3** (Multi-Segre varieties)**.** We can also consider Segre embeddings with more factors than two: for example, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{15}$. Let $\mathcal{V}_i$ be the variety in $\mathrm{Gr}(2, 16)$ corresponding to embedding the $i$th factor, and letting points on the other factors vary. In other words, $\mathcal{V}_i$ is a family of lines parameterized by $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $\dim \mathcal{V}_i = 3$, and so $\mathcal{V}_i$ has a Chow–Lam locus which lives in $\mathrm{Gr}(16 - 4 + 3, 16) = \mathrm{Gr}(15, 16)$. It is exactly the dual variety to the Segre variety.

More generally, for any $k \geq 2$, $i \geq 0$, the Segre embedding $(\mathbb{P}^{k-1})^{ki+2}$ is ruled by $ki + 2$ families of $(k-1)$-spaces. This gives an example of $ki + 2$ varieties in $\mathrm{Gr}(k, N)$ with the same Chow–Lam locus, where $N = k^{ki+2} - 1$. Each variety has dimension $(ki + 1)(k - 1) = k(r - k) - 1$, where $r = ki + 1 - i + k$. By [Gel+94, Theorem 0.1] the dual variety of the Segre embedding is non-degenerate. Thus the Chow–Lam form of each $\mathcal{V}_i$ equals this variety.  ◇

In the previous examples, the ambient Grassmannian $\mathrm{Gr}(k, n)$ has $n$ equal to a power of $k$. We now give a family of examples in $\mathrm{Gr}(k, 2k)$ of varieties with the same Chow–Lam form, thereby demonstrating that the failure of unique recovery can also occur in smaller Grassmannians. Fix an integer $k \geq 2$. We identify $\mathbb{P}^{k^2-1}$ with the space of $k \times k$ matrices, and use $S_k$ to denote the hypersurface in $\mathbb{P}^{k^2-1}$ cut out by the determinant. This has two rulings $\mathcal{V}_1, \mathcal{V}_2$ by linear spaces of codimension $k$, which we explain as follows. For $i \in [k]$, let $R_i(x)$ denote the $i$th row of the $k \times k$ matrix $x$. Then $R_i(x)$ is a vector of length $k$. Consider the map

$$\begin{aligned}
\mathbb{P}^{k-1} &\to \mathrm{Gr}(k, k^2) \\
[a_1 : \ldots : a_k] &\mapsto \{x \in \mathbb{P}^{k^2-1} \ : \ \sum a_1 R_1(x) + \cdots + a_k R_k(x) = 0\}.
\end{aligned} \tag{6.5}$$

The last equality is an equality of vectors, so each point $[a_1 : \cdots : a_k]$ yields a linear space of codimension $k$.

The dual variety of $S_k$ is the Segre variety $\Sigma_k$ [GKZ94, Proposition 1.4.11]; thus $S_k$ has a degenerate dual, and we cannot apply Corollary 6.4.1. However, we may remedy the situation by taking a general linear section. The resulting Chow–Lam form will have degree $\binom{2k-2}{k-1}$, which is the degree of the Segre variety.

**Proposition 6.4.4** (Chow–Lam forms from determinantal varieties)**.** *Fix an integer $k \geq 2$ and a general linear space $L$ in $\mathbb{C}^{k^2}$ of dimension $2k$. Then the determinantal variety*

$D_k := \mathbb{P}L \cap S_k$ *has two rulings of* $(k-1)$*-spaces. These two rulings yield two subvarieties in* $\mathrm{Gr}(k, L) \cong \mathrm{Gr}(k, 2k)$ *whose Chow–Lam loci coincide, and equal* $\mathrm{Proj}_L(\Sigma_k) \subset \mathbb{P}(L)^\vee$.

*Proof.* Note that $\mathbb{P}L^\perp$ has dimension $k^2 - 2k + 1$, whereas the Segre variety $\Sigma_k$ has dimension $2k - 2$. Thus they will not intersect if $L$ is general. This means the projection of $\Sigma_k$ to $L$ does not have a base locus. In this case, the dual variety of a linear section $\mathbb{P}L \cap S_k$ coincides with the projection $\mathrm{Proj}_{\mathbb{P}L}(\Sigma_k)$ [GKZ94, Proposition 1.4.1]. By a dimension count, the latter is a hypersurface. Thus $D_k$ has a non-degenerate dual, and we may apply Corollary 6.4.1. The two subvarieties of $\mathrm{Gr}(k, L)$ are the intersections of the varieties $\mathcal{V}_1, \mathcal{V}_2$ with $\mathrm{Gr}(k, L)$ inside of $\mathrm{Gr}(k, k^2)$. □

We illustrate Proposition 6.4.4 with the concrete example of $k = 3$. In this case we uncover a connection to lines on a cubic surface.

**Example 6.4.5** (Cubic surface)**.** Consider a $3 \times 3$ matrix $M$ filled with linear forms in six variables. Let $D_3$ denote the determinantal hypersurface cut out by $\det M$. By Proposition 6.4.4 and the discussion preceding it, we have that $D_3$ is ruled by two families $\mathcal{V}_1, \mathcal{V}_2$ of planes. Concretely, let $R_1(x), R_2(x)$, and $R_3(x)$ be the three-dimensional vectors of linear forms coming from the rows of $M$. Consider the map

$$\mathbb{P}^2 \to \mathrm{Gr}(3, 6)$$
$$[a : b : c] \mapsto \{x \in \mathbb{P}^5 \ : \ aR_1(x) + bR_2(x) + cR_3(x) = 0\}.$$

This gives a subvariety $\mathcal{V}_1$ of $\mathrm{Gr}(3, 6)$ which rules $D_3$ with planes. One repeats the construction with columns to obtain another ruling $\mathcal{V}_2$. The common Chow–Lam locus of $\mathcal{V}_1$ and $\mathcal{V}_2$ is the projection of the Segre variety $\Sigma_3$ from $(\mathbb{P}^8)^\vee$ to $(\mathbb{P}^5)^\vee$. Its degree is $\binom{2 \cdot 3 - 2}{3 - 1} = 6$.

There is another way to interpret this six. To obtain the degree of $D_3^\vee$, we intersect it with a line in $(\mathbb{P}^5)^\vee$. This is equivalent to intersecting $S$ with a general pencil $\mathcal{L}$ of hyperplanes in $\mathbb{P}^5$. The base locus $L$ of this pencil, namely the intersection of all hyperplanes in the pencil, has dimension three. The six intersection points $H_1, \ldots, H_6$ of $D_3^\vee$ with $\mathcal{L}$ lie in the common Chow–Lam locus of $\mathcal{V}_1$ and $\mathcal{V}_2$. Thus each of the six hyperplanes $H_i$ contains two planes $P_i$ and $P_i'$ in $D_3^\vee$. Moreover, $H_i$ contains $L$, so $P_i$ and $P_i'$ each meet $L$ in a line.

The linear section $D_3^\vee \cap L$ is a cubic surface in $L \cong \mathbb{P}^3$. It is well-known that every smooth cubic surface contains exactly 27 distinct lines. The pairs of planes $(P_i, P_i')$ give two families of six lines on the cubic surface, which form a "double six"; namely, they do not pairwise intersect and any line from one family meets exactly five lines from the other family. ◇

## 6.5 Schubert arrangements

In this section we explore in detail the case where $\mathcal{V}$ is a linear section of $\mathrm{Gr}(k, n)$ by Schubert divisors. In this case the variety $W_\mathcal{V}$ contains additional linear components not in $\mathcal{V}$.

Let $H$ be a linear space of codimension $k$ in $\mathbb{P}^{n-1}$. Let $\Omega_1(H) \subset \mathrm{Gr}(k, n)$ be the variety of subspaces in $\mathbb{P}^{n-1}$ which meet $H$. In our notation, $\Omega_1(H)$ is isomorphic to the Schubert

variety $\Omega_1$, i.e. the Schubert variety of codimension 1 in $\mathrm{Gr}(k,n)$. In this section, we will consider Schubert varieties of the form $\Omega_1^j = \Omega_1(H_1) \cap \cdots \cap \Omega_1(H_j)$, for $j$ general linear spaces $H_i$, with $j = k(n-r)+1$ for some $k < r < n$. We will refer to the Schubert varieties in $W_{\mathcal{V}} \setminus \mathcal{V}$ as *recovered Schubert components*. Note that our notation for $\Omega_1^j$ does not depend on $n$; we will show that given $j$, there is a set of recovered Schubert components which will appear for any $n$ large enough.

**Example 6.5.1** (Threefold in $\mathrm{Gr}(2,5)$). Choose $H_1, H_2, H_3$ general planes in $\mathbb{P}^4$. Consider the threefold $\mathcal{V} = \Omega_1^3$ of lines in $\mathbb{P}^4$ meeting all three planes. The Chow–Lam locus of $\mathcal{V}$ will live in $\mathrm{Gr}(3,5)$. Then $W_{\mathcal{V}}$ is the variety of all lines $L$ such that for all planes $Q$ containing $L$, $Q$ also contains a common transversal to $H_1, H_2$, and $H_3$. We claim that $W_{\mathcal{V}}$ also contains three extra components of each of two equivalence classes:

i) The Schubert variety $\Omega_3$ of lines meeting $H_i \cap H_j$

ii) The Schubert variety $\Omega_{2^2}$ of lines contained in $H_i$.

Let $Q \cong \mathbb{P}^2$ be any plane. For a generic choice of $Q$, the planes $H_1, H_2, H_3$ will intersect $Q$ in three points $p_1, p_2, p_3$, which do not have a common transversal. To show that $L$ is in the recovery, we need to check that the condition $Q \supset L$ forces $p_1, p_2, p_3$ to have a common transversal in $Q$.

For case (i), suppose $L$ meets $H_1 \cap H_2$ and consider the geometry in a generic plane $Q \cong \mathbb{P}^2$ containing $L$. Then $Q$ also contains the point $p = H_1 \cap H_2 \in L$. Thus the line $\overline{pp_3}$ is a common transversal to $H_1, H_2, H_3$.

For case (ii), suppose $L$ is contained in $H_1$ and consider a generic plane $Q$ containing $L$. Then $H_1$ intersects $Q$ not in a point, but a line: the line $L$ itself. Then the line $\overline{p_2 p_3}$ will intersect $L$ in $Q$ by Bézout's theorem. Thus $\overline{p_2 p_3}$ will be a common transversal to $H_1, H_2, H_3$. Both these cases are illustrated in Figure 6.2.

Finally, we note that $\mathcal{V} = \Omega_1^3 \subset \mathrm{Gr}(2,n)$ will have these recovered components for all $n \geq 5$, since nothing about the geometry in $Q$ uses the value of $n$. ◇

**Example 6.5.2** (Codimension seven variety). Similarly, $\Omega_1^7 \subset \mathrm{Gr}(2,n)$ will have recovered Schubert components of two classes: $\binom{7}{3}$ of type $\Omega_{6^2}$ (lines contained in $H_i \cap H_j \cap H_k$) and $\binom{7}{4}$ of type $\Omega_7$ (lines meeting $H_i \cap H_j \cap H_k \cap H_l$).

To see this, we observe that the Chow–Lam locus lives in $\mathrm{Gr}(5,n)$. Consider a general subspace $Q$ in the Chow–Lam locus. Then $Q$ has dimension 4 and contains seven planes $P_i := Q \cap H_i$. A generically chosen $Q$ will not contain a transversal to $P_1, ..., P_7$. We would like to show that if $Q$ is *specially* chosen to contain a line $L$ in one of the Schubert classes above, then $Q$ must also contain a transversal line $L'$ to the seven $P_i$.

For an example of the first type, consider the case $n = 8$ and let $L$ denote the line $H_1 \cap H_2 \cap H_3$. Suppose $Q$ contains $L$. Then any line $L'$ in $Q$ which meets $L$ and $P_4, ..., P_7$ will be a common transversal to the seven planes. The subvariety of such $L'$ in $\mathrm{Gr}(2,5)$ has cohomology class $[\Omega_2][\Omega_1]^4 \neq 0$, and thus a common transversal must exist.

$L$ is in $\Omega_3$          $L$ is in $\Omega_{2,2}$
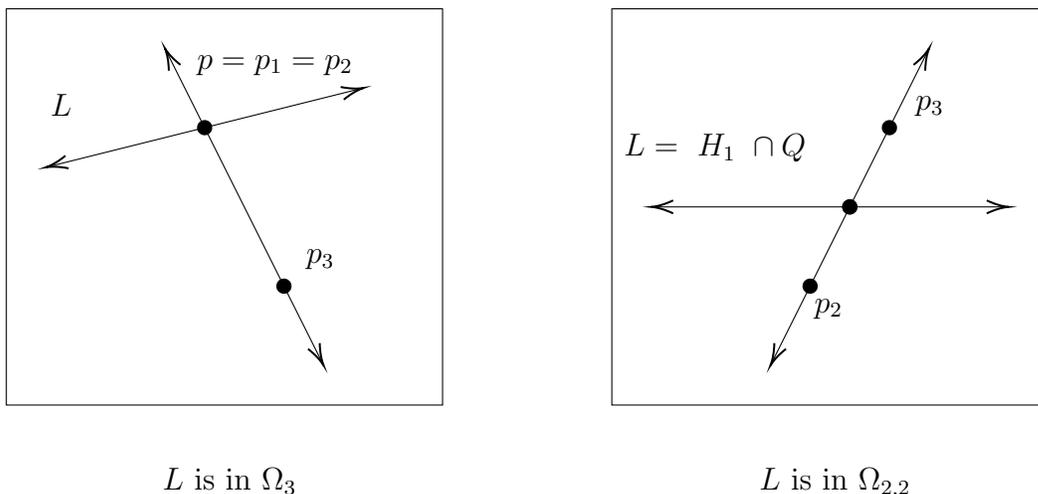
Figure 6.2: Geometry in $Q \cong \mathbb{P}^2$

For an example of the second type, consider the case $n = 9$ and a line $L$ that meets the point $H_1 \cap \cdots \cap H_4$. Suppose $Q$ contains $L$, and thus contains the intersection of the four hyperplanes. Then any line $L'$ in $Q$ through the point $H_1 \cap \cdots \cap H_4$ and meeting $P_5, P_6, P_7$ will be a common transversal. This has class $[\Omega_3][\Omega_1]^3$, which is nonzero in $H^*(\mathrm{Gr}(2,5))$.

Note that we have conditions on $n$ in both cases for these components to actually appear for generic $H_i$. For the class $\Omega_{6^2}$, we must have that the three $H_i$ intersect in at least a line in $\mathbb{P}^{n-1}$, so that $L$ can be contained in the intersection. Thus we want $(n-1) - \mathrm{codim}\, H_1 \cap H_2 \cap H_3 = (n-1) - 3 \cdot 2 \geq 1$, meaning that $n$ is at least 8. For the second class, we want that $(n-1) - \mathrm{codim}\, H_1 \cap H_2 \cap H_3 \cap H_4 \geq 0$, so $n \geq 9$. Also, $n$ must be at least 8 and 9 for $\Omega_{6^2}$ and $\Omega_7$ to yield nonzero classes in the cohomology of $\mathrm{Gr}(k,n)$. $\diamond$

It turns out that $\Omega_1^{ki+1} \subset \mathrm{Gr}(k,n)$ will have additional recovered Schubert components for all $i$, provided that $n$ is sufficiently high. Here we list some recovered components for small $k$ and $i$, assuming that $n$ is higher than a lower bound described in Theorem 6.5.3.

**Theorem 6.5.3.** *Fix $k$ and $i > k$. Consider $\mathcal{V} := \Omega_1^{ki+1} \subset \mathrm{Gr}(k,n)$ for $n > k(i+1)+1$. Then the Chow–Lam recovery $W_{\mathcal{V}}$ will contain recovered components of $k$ distinct Schubert types $\Omega_{ki+1}, \Omega_{(k(i-1)+2)^2}, \Omega_{(k(i-2)+3)^3}, ..., \Omega_{(k(i-k+1)+k)^k}$.*

*Proof.* Fix $a$ between 0 and $k-1$, and fix any $i+1-a$ of the $ki+1$ codimension $k$ spaces $H_i$. Let $H$ denote their intersection of codimension $k(i+1-a)$ in $\mathbb{P}^{n-1}$. Consider the Schubert variety of linear spaces of projective dimension $k-1$ which intersect $H$ in dimension at least $a$. This is isomorphic to $\Omega_{(k(i-a)+a+1)^{a+1}}$. We will show that this Schubert variety is a recovered component of $\Omega_1^{ki+1}$, provided that $a$ is sufficiently large.

| $k$ | $i$ | $\mathcal{V}$ | Recovered | Interpretation, with $\cap_i H := H_1 \cap \ldots \cap H_i$ | Lower bounds on $n$ |
|---|---|---|---|---|---|
| 2 | 3 | $\Omega_1^7$ | $\Omega_7, \Omega_{6^2}$ | line meets $\cap_4 H$, line contained in $\cap_3 H$ | 9, 8 |
| 2 | 4 | $\Omega_1^9$ | $\Omega_9, \Omega_{8^2}$ | line meets $\cap_5 H$, line contained in $\cap_4 H$ | 11, 10 |
| 2 | $i$ | $\Omega_1^{2i+1}$ | $\Omega_{2i+1}, \Omega_{(2i)^2}$ | line meets $\cap_{i+1} H$, line contained in $\cap_i H$ | $2i+3$, $2i+2$ |
| 3 | 4 | $\Omega_1^{13}$ | $\Omega_{13}, \Omega_{11^2}, \Omega_{9^3}$ | plane meets $\cap_5 H$, plane meets $\cap_4 H$ in a line, plane contained in $\cap_3 H$ | 16, 14, 12 |
| 3 | $i$ | $\Omega_1^{3i+1}$ | $\Omega_{3i+1}, \Omega_{(3i-1)^2}, \Omega_{(3i-3)^3}$ | plane meets $\cap_{i+1} H$, plane meets $\cap_i H$ in a line, plane contained in $\cap_{i-1} H$ | $3i+4$, $3i+2$, $3i$ |

Table 6.1: Some recovered components for some small values of $k$ and $i$

Choose a $(k-1)$-space $P$ intersecting $H$ in at least dimension $a$. Such a space $P$ exists, because we assumed $n \geq k(i+1)+1$. This implies

$$n - 1 - \operatorname{codim}\ H \geq a,$$

where codim $H = k(i+1-a)$. Thus $P$ is in the Schubert variety $\Omega_{(k(i-a)+a+1)^{a+1}}$.

Note that the Chow–Lam locus of $\Omega_1^{ki+1}$ is a subset of $\operatorname{Gr}(i+k, n)$. Choose a generic space $Q \cong \mathbb{P}^{i+k-1} \subset \mathbb{P}^{n-1}$ containing $P$. Let $P_1, ..., P_{ki+1}$ be the intersections of the codimension $k$-spaces $H_1, ..., H_{ki+1}$ with $Q$. We will show that the $P_i$ have a common transversal $P' \in \operatorname{Gr}(k, \hat{Q}) = \operatorname{Gr}(k, i+k)$ in $Q$. This implies that $P$ is in the Chow–Lam recovery of $\mathcal{V}$.

If $Q$ contains $P$, then $Q$ also intersects $H$ in at least dimension $a$ and any $(k-1)$-space $P'$ in $Q$ meeting $H$ and the remaining $P_i$ is a common transversal. The class of such $P'$ is $\Omega_{i-a} \cdot \Omega_1^{ki+1-(i+1-a)}$, which is nonzero in the cohomology ring of $\operatorname{Gr}(k, i+k)$. Thus there exists a common transversal. $\qquad\square$

We caution that not all recovered components of $\Omega_1^{ki+1} \subset \operatorname{Gr}(k, n)$ are of the form in Theorem 6.5.3. For example, the Schubert variety $\Omega_{2,2} \cap \Omega_3$ of lines contained in $H_1$ and intersecting $H_2 \cap H_3$ is in the recovered locus of $\Omega_1^5 \subset \operatorname{Gr}(2, 7)$. Indeed, consider a line $L$ in this Schubert variety. In a generic 3-space $Q \cong \mathbb{P}^3$, there are five lines $L_i := H_i \cap Q$. If $Q$ contains $L$, then $L_1 = L$ intersects $H_2 \cap H_3$ at a point $p$. Then $p$ must be in $Q$, so it is also the intersection point of the lines $L_2$ and $L_3$. Thus there are common transversals of the five lines: those which pass through the point $p$ and intersect the lines $L_4, L_5$, of which there is one since $[\Omega_2] \cdot [\Omega_1]^2 = 1$ in $\operatorname{Gr}(2, \hat{Q})$. This is shown in Figure 6.3. However, this case is

special because $i$ is small, which allows for the possibility that $L_i$ is equal to $L$; for $i$ large, $H_i \cap Q$ will have dimension higher than $k$.
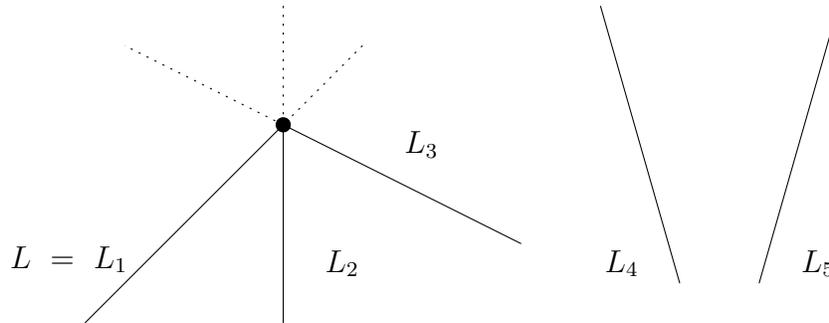


Figure 6.3: Geometry of the lines $L_i$ in $Q \cong \mathbb{P}^3$

**Conjecture 6.5.4.** If $i \geq k$ then all recovered components of $\Omega_1^{ki}$ are of the form in Theorem 6.5.3.

One could also ask what happens when a linear section of $\mathrm{Gr}(k, n)$ is not the intersection of Schubert hyperplane sections: do generic $(k(r-k)-1)$-dimensional linear sections of the Grassmannian have residual components in their Chow–Lam recovery? This is difficult to establish in general. The special case of curves may indicate what to expect. Let $\mathcal{V}$ be the intersection of $\mathrm{Gr}(2, n)$ with a general linear space of codimension $2n-5$ in $\mathbb{P}^{\binom{n}{2}-1}$. Then $\mathcal{V}$ is a curve, and if $n \neq 4$, there is no line $L$ in $X_{\mathcal{V}}$ that does not belong to $\mathcal{V}$, i.e. $W_{\mathcal{V}} = \mathcal{V}$. This is because $X_{\mathcal{V}}$ is smooth and therefore contains a line not in the ruling only if $\mathcal{V}$ is rational. But, by adjunction, the canonical divisors on $\mathcal{V}$ are the restriction of hypersurfaces of degree $2n - 5 - n = n - 5$, so $\mathcal{V}$ is rational only if $n = 4$. We note that in this case $\mathcal{V}$ must have degree 2.

A threefold example shows that the analysis in general may be more involved. In it we are able to prove there are no residual components, using the tools developed in Section 6.2.

**Proposition 6.5.5.** *Let $Q$ be the intersection of three generic hyperplanes in the Plücker space $\mathbb{P}^9$ of $\mathrm{Gr}(2, 5)$, and let $\mathcal{V} = Q \cap \mathrm{Gr}(2, 5)$. Then $W_{\mathcal{V}}$ is equal to $\mathcal{V}$, i.e. has no residual components.*

*Proof.* Let $\mathcal{V}$ be $Q \cap \mathrm{Gr}(2, 5)$. Then $\mathcal{V}$ is a threefold and $r = 4$. Suppose there is a recovered line $L$. By Theorem 6.2.2, we must have that the dimension of the space $\mathcal{V}_0$ of lines in $\mathcal{V}$ meeting $L$ is at least two.

Since $Q$ is generic, the variety $\mathcal{V}$ is a smooth Fano threefold of degree 5. In particular its Picard group has rank one and is generated by the class of a hyperplane section (cf. [Isk77, Corollary 6.6]). Note that $\mathcal{V}_0$ is a divisor in $\mathcal{V}$. Thus its class is some multiple of the hyperplane class in $\mathcal{V}$.

We can describe $\mathcal{V}_0$ even more explicitly. Let $T_L$ denote the embedded tangent space $T_L \operatorname{Gr}(2,5)$ at the point $L$. The variety $\Omega_3 = T_L \cap \operatorname{Gr}(2,5)$ is the four-dimensional Schubert variety of lines in $\mathbb{P}^4$ meeting $L$. Indeed, let $L$ be the rowspan of a matrix $A$ and take a general line $M$ given as the rowspan of $B$. Then the expression $\operatorname{Jac}_L \operatorname{Gr}(2,5) \cdot (B_{ij})$ in dual Plücker coordinates is equal to zero whenever all the $2 \times 2$ minors of the matrix obtained by stacking $A$ and $B$ vanish. Thus we have a containment

$$\mathcal{V}_0 \subset Q \cap T_L \operatorname{Gr}(2,5) \cap \operatorname{Gr}(2,5) = \Omega_3.$$

Since $\mathcal{V}$ is smooth, the linear space $Q$ does not contain $[L]$. Thus the containment is strict, and $\mathcal{V}_0$ is a surface in this threefold.

Since $\Omega_3$ is a linear space, it is its own embedded tangent space at any smooth point $M$ of $\Omega_3$. By a standard description of tangent varieties to Schubert varieties, $T_M \Omega_3$ may be identified with the space of homomorphisms from $L$ to $(M + L)/L$, which may be further identified with the space of rank $2 \times 3$ matrices. Thus $\Omega_3$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ in $\mathbb{P}^6$, where $\mathbb{P}^6$ is isomorphic to the projective closure of $T_L \operatorname{Gr}(2,5)$ in $\operatorname{Gr}(2,5)$. Since $\mathcal{V}_0$ is a surface in this intersection, it has degree at most 3. But the degree of every surface in $\mathcal{V}$ is divisible by five, so this is a contradiction. $\qquad \square$

Based on these first examples we boldly conjecture:

**Conjecture 6.5.6.** Fix $r$, with $k < r \leq n$. Let $\mathcal{V}$ be the intersection of $\operatorname{Gr}(k,n) \subset \mathbb{P}^N$ with a generic $(k(n-r) + 1)$-dimensional linear space in $\mathbb{P}^N$. Then the algebraic sets $W_{\mathcal{V}}$ and $\mathcal{V}$ coincide.

Finally, we return to the study of positroid varieties laid out in the introduction and in Section 5.3. Understanding projections of these varieties would aid in understanding the amplituhedron itself. The class of examples in Theorem 6.5.3 includes certain positroid varieties, as some of them can be cut out by generic Schubert divisors.

**Corollary 6.5.7.** Let $n = 2i + 1$ be odd and at least three. Consider the positroid variety $\mathcal{V}$ in $\operatorname{Gr}(2, 2n)$ given by the vanishing of $p_{12}, ..., p_{2n-1,2n}$. Then $W_{\mathcal{V}} \neq \mathcal{V}$.

*Proof.* Each condition of the form $p_{i,i+1} = 0$ defines a Schubert hyperplane. Thus $\mathcal{V}$ is of the form $\Omega_1^n$, and we only need to check the bounds in Theorem 6.5.3. But $2n > 2(i+1)+1 = n+2$ whenever $n$ is greater than two. $\qquad \square$

It would be interesting to undertake a systematic study of $W_{\mathcal{V}}$ where $\mathcal{V}$ is a positroid variety, using the tools developed in this chapter.

# Chapter 7

# The Segre Determinant

This chapter is based on the single-author paper [Pra25]. It is mostly the same, but Section 7.5 has now been expanded. and Section 7.6 has been added.

## 7.1 Definitions and first examples

Fix vector spaces $V$ and $W$ over $\mathbb{C}$ of dimensions $k$ and $\ell$. Consider $r$ points $A_1 \times B_1$, ..., $A_r \times B_r$ in $\mathbb{P}(V) \times \mathbb{P}(W)$. The *Segre matrix* of this point configuration is the $k\ell \times r$ matrix

$$\begin{bmatrix} \vdots & & \vdots \\ A_1 \otimes B_1 & \dots & A_r \otimes B_r \\ \vdots & & \vdots \end{bmatrix}. \tag{7.1}$$

It is a flattening of the $k \times \ell \times r$ tensor with slices $A_i \otimes B_i$. When $r = k\ell$, the Segre matrix is square and we call its determinant the *Segre determinant*. This determinant vanishes whenever the $k\ell$ points lie on a hyperplane section of $\mathbb{P}(V) \times \mathbb{P}(W)$ in the Segre embedding, hence the name. Indeed, the equation of such a hyperplane section is in the left kernel of the Segre matrix. We will use $\mathrm{Seg}_{k,\ell}$ to denote the Segre determinant as a polynomial in the coordinates of the points $A_i$ and $B_i$.

The aim of this chapter is to study the Segre determinant and give various applications. After establishing its basic properties in Section 7.2, we prove our main result in Section 7.3:

**Theorem 7.3.2** (Segre Determinant)**.** Suppose $k, \ell \geq 2$ and let $n = k\ell$. Fix a point $A$ in $\mathrm{Gr}(k, n)$ with non-zero Plücker coordinates. Then the Chow–Lam form of the torus orbit of $A$ in primal Plücker coordinates $B$ on $\mathrm{Gr}(n-\ell, n)$ equals the Segre determinant $\mathrm{Seg}_{k,\ell}(A, B)$.

The Segre determinant is a generalization of the classical determinant for $k = 1$. The special case $k = \ell = 2$ encodes the condition for two configurations of four ordered points in $\mathbb{P}^1$ to be the same up to automorphism; see Example 7.1.1 below. The case $k = \ell = 3$ appears in algebraic vision as a necessary condition for two configurations of nine ordered points in $\mathbb{P}^2$ to be linear projections of a common configuration of ordered points in $\mathbb{P}^3$ [Aga+16].

**Example 7.1.1.** Consider four points $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \times \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1,4} \\ a_{2,4} \end{bmatrix} \times \begin{bmatrix} b_{1,4} \\ b_{2,4} \end{bmatrix}$ in the surface $\mathbb{P}^1 \times \mathbb{P}^1$. The Segre matrix looks like

$$\begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\ a_{1,1}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,3} & a_{1,4}b_{2,4} \\ a_{2,1}b_{1,1} & a_{2,2}b_{1,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,4} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \end{bmatrix}. \tag{7.2}$$

This expression vanishes whenever the four points in $\mathbb{P}^1 \times \mathbb{P}^1$ lie on a common $(1,1)$-curve. We collect our eight vectors into two matrices

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}, \qquad B := \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{bmatrix}.$$

Let $[ij]$ denote the six maximal minors of $A$ and $\langle ij \rangle$ denote the six maximal minors of $B$. Then we compute

$$\mathrm{Seg}_{2,2} = [12][34]\langle 13\rangle\langle 24\rangle - [13][24]\langle 12\rangle\langle 34\rangle.$$

It vanishes whenever the cross-ratio $\frac{[12][34]}{[13][24]}$ of the four points $A_i$ in $\mathbb{P}^1$ is equal to the cross-ratio of the four points $B_i$ in $\mathbb{P}^1$. The cross-ratio is the fundamental invariant for $\mathrm{PGL}_2$ acting on $(\mathbb{P}^1)^4$. $\diamond$

The next two sections focus on computations and examples. In Section 7.2 we prove that the Segre determinant may be written in terms of the maximal minors of the $k \times k\ell$ matrix $A$ and the $\ell \times k\ell$ matrix $B$, generalizing the computation in Example 7.1.1. These minors generate the invariant rings of $\mathrm{SL}(k)$ and $\mathrm{SL}(\ell)$ acting on $(\mathbb{P}^{k-1})^n$ and $(\mathbb{P}^{\ell-1})^n$, respectively [Stu93, Theorem 3.2.1].

We also discuss how to compute such expressions, which is difficult in practice. This flavor of question has a long history in classical algebraic geometry, and is related to the problem of synthetic (ruler-and-compass) constructions. The analogous problem for the Veronese embedding is to figure out when $\binom{n+k-1}{k}$ points in $\mathbb{P}^{n-1}$ lie on a hypersurface of degree $k$. Pascal's theorem for $k = 2$ and $n = 3$ is the earliest such result, giving a condition for six points in the plane to lie on a conic. The *Bruxelles problem*, posed in 1825 by l'Académie de Bruxelles, asks for a synthetic construction to determine when ten points in $\mathbb{P}^3$ lie on a quadric surface ($k = 2$, $n = 4$). This is a significant leap in difficulty, and a construction was only recently obtained by Traves [Tra24].

Such a property of a point configuration is $\mathrm{SL}(k)$-invariant, so we may ask how to write it in terms of minors of $A$ and $B$, perhaps thus gleaning some geometric insight. Indeed, in 1927 Turnbull and Young gave an algebraic expression in terms of the $4 \times 4$ minors of the $4 \times 10$ matrix parameterizing the points [TY27]. In this spirit, we present the analogous computation for eight points in $\mathbb{P}^1 \times \mathbb{P}^3$ and for nine points in $\mathbb{P}^2 \times \mathbb{P}^2$ in Equations (7.9) and (7.12), respectively.

Section 7.3 states and proves our main result, Theorem 7.3.2. Finally, we close this chapter with two other applications of the Segre determinant. Section 7.4 gives an application of the Segre determinant $\text{Seg}_{3,3}$ to algebraic vision. In this setting, the Segre determinant is a necessary condition for two distinct configurations of nine ordered points in $\mathbb{P}^2$ to be linear projections of a common set of nine ordered points in $\mathbb{P}^3$. Our computation answers a question of Rekha Thomas about writing this condition in terms of $\text{SL}_3$-invariants. This section is completely independent from Section 7.3.

In Section 7.5 we define the *Segre coefficient variety*. It parameterizes Segre determinants as polynomials in the $B$ variables as $A$ varies. We show in Theorem 7.5.2 that every monomial of the form $[I_1] \cdots [I_\ell]$ with $I_1 \cup \ldots \cup I_\ell = [k\ell]$ appears in the linear span of the coefficients of the Segre polynomial. This results in Corollary 7.5.3:

**Theorem 7.5.3.** The variety $\text{Coeff}(\text{Seg}_{2,\ell})$ is isomorphic to the GIT quotient $(\mathbb{P}^1)^{2\ell} /\!/_w \text{SL}_2$, where $w = 1^{2\ell}$ is the linearization.

Thus the Segre determinant uniquely determines torus orbit closures of generic points in $\text{Gr}(2, n)$. However, this fails for $k = 3$. We give two torus orbit closures with the same Segre determinant in Example 7.5.4.

# 7.2 Properties and computation

We now discuss the computation of the Segre determinant in various coordinate systems. As in the introduction, fix vector spaces $V$ and $W$ of dimensions $k$ and $\ell$, and let $n := k\ell$.

Consider the polynomial rings $\mathbb{C}[a_{ij}]_{1 \leq i \leq k, 1 \leq j \leq n}$ and $\mathbb{C}[b_{ij}]_{1 \leq i \leq \ell, 1 \leq j \leq n}$. As in the introduction, we may collect the indeterminates $a_{ij}$ and $b_{ij}$ into a $k \times n$ matrix $A$ and an $\ell \times n$ matrix $B$. Given a subset $I \in \binom{[n]}{k}$, we define the *bracket* $[I]$ to be the determinant of the submatrix of $A$ with columns indexed by $I$. Similarly, given $J \in \binom{[n]}{\ell}$, we define the bracket $\langle J \rangle$ to be the determinant of the submatrix of $B$ with columns $J$. These rings have actions of $\text{SL}(k)$ and $\text{SL}(\ell)$, respectively, given by left multiplication of the matrices $A$ and $B$. The following is sometimes called the *First Fundamental Theorem of Invariant Theory.*

**Theorem 7.2.1** (Theorem 3.2.1 of [Stu93])**.** *The brackets* $[I]$ *generate the invariant ring* $\mathbb{C}[a_{ij}]^{\text{SL}(k)}$.

We denote the invariant ring $\mathbb{C}[a_{ij}]^{\text{SL}(k)}$ by $\mathcal{B}_{k,n}$ and call it the *bracket algebra*. The Segre determinant lives in the tensor product $\mathbb{C}[a_{ij}] \otimes \mathbb{C}[b_{ij}]$ of $\mathbb{C}$-algebras. It is separately invariant under the $\text{SL}(k)$ and $\text{SL}(\ell)$ actions on $V \otimes W$, and thus may be written either in brackets $[I]$ and indeterminates $b_{ij}$, or in brackets $\langle J \rangle$ and indeterminates $a_{ij}$. These can be realized concretely as two different block Laplace expansions of the Segre matrix. The following result states that $\text{Seg}_{k,\ell}$ may be written *simultaneously* in the two systems of brackets.

**Proposition 7.2.2.** *The Segre determinant* $\text{Seg}_{k,\ell}$ *is a polynomial of bi-degree* $(\ell, k)$ *in the brackets* $[I]$ *and* $\langle J \rangle$.

The main ingredient in proving Proposition 7.2.2 is the following lemma.

**Lemma 7.2.3.** *Suppose that $V$ and $W$ are finite-dimensional representations of groups $G$ and $H$, respectively. Then*

$$V^G \otimes W^H = (V \otimes W)^{G \times H}. \tag{7.3}$$

*Proof.* The inclusion $\subseteq$ is immediate. For the other inclusion, choose a basis $\alpha_1, \ldots, \alpha_r$ for $V^G$ and add vectors $v_j$ to extend it to a basis for $V$. Similarly, choose a basis $\beta_1, \ldots, \beta_s$ for $W^H$ and add vectors $w_j$ to extend it to a basis for $W$. We observe that the right-hand side of (7.3) is contained in the vector space

$$(V^G \otimes W) \cap (V \otimes W^H) \subseteq V \otimes W.$$

An element $f$ of the intersection may be written uniquely in each basis as

$$f = \sum_{i \leq r, j \leq s} c_{ij} \alpha_i \otimes \beta_j + \sum_{i \leq r, j > s}^{k} c_{ij} \alpha_i \otimes w_j \tag{7.4}$$

$$= \sum_{i \leq r, j \leq s} d_{ij} \alpha_i \otimes \beta_j + \sum_{i > r, j \leq s}^{k} d_{ij} v_i \otimes \beta_j. \tag{7.5}$$

Since the expressions are unique, we must have that $f = \sum_{i \leq r, j \leq s} c_{ij} \alpha_i \otimes \beta_j$. $\qquad \square$

*Proof of Proposition 7.2.2.* The proof is mostly type-checking. For clarity, within this proof we use $\boxtimes$ when forming the representation of a direct product of groups. The Segre determinant lives in the coordinate ring $\mathrm{Sym}^\bullet(V \boxtimes W) \otimes \cdots \otimes \mathrm{Sym}^\bullet(V \boxtimes W)$ of the product $\mathbb{P}(V \boxtimes W) \times \cdots \times \mathbb{P}(V \boxtimes W)$. This ring has a multi-grading given by taking the grading in each tensor factor, and the Segre determinant lives in the component with grading $1^n = (1, \ldots, 1)$.

Consider the map of $\mathrm{SL}(k) \times \mathrm{SL}(\ell)$-representations

$$\psi : (V \boxtimes W)^{\otimes n} \to V^{\otimes n} \boxtimes W^{\otimes n}$$
$$(x_1 \boxtimes y_1) \otimes \ldots \otimes (x_n \boxtimes y_n) \mapsto (x_1 \otimes \ldots \otimes x_n) \boxtimes (y_1 \otimes \ldots \otimes y_n). \tag{7.6}$$

There is a map $\mathrm{SL}(k) \times \mathrm{SL}(\ell) \to \mathrm{GL}(V \boxtimes W)$ given by $(A, B) \cdot x_i \otimes y_j = Ax_i \otimes By_j$. The image lies in $\mathrm{SL}_{k\ell} = \mathrm{SL}(V \boxtimes W)$. Thus there is an inclusion of invariant vector spaces

$$\iota : \left((V \boxtimes W)^{\otimes n}\right)^{\mathrm{SL}_{k\ell}} \hookrightarrow (V^{\otimes n} \boxtimes W^{\otimes n})^{\mathrm{SL}(k) \times \mathrm{SL}(\ell)}. \tag{7.7}$$

Finally, by Lemma 7.2.3 we have that

$$(V^{\otimes n} \boxtimes W^{\otimes n})^{\mathrm{SL}(k) \times \mathrm{SL}(\ell)} = (V^{\otimes n})^{\mathrm{SL}(k)} \boxtimes (W^{\otimes n})^{\mathrm{SL}(\ell)}. \tag{7.8}$$

By the First Fundamental Theorem of Invariant Theory (Theorem 7.2.1), the right-hand side of (7.8) is the algebra generated by the brackets $[I]$ and $\langle J \rangle$. By (**??**) the Segre determinant $\mathrm{Seg}_{k,\ell}$ lies in $(V^{\otimes n} \boxtimes W^{\otimes n})^{\mathrm{SL}(k) \times \mathrm{SL}(\ell)}$, and applying $\iota$ gives its expansion into brackets $[I]$ and $\langle J \rangle$. Note that the total degree of $\mathrm{Seg}_{k,\ell}$ in each set of variables $a_{ij}$ and $b_{ij}$ is $n$. Since $[I]$ has degree $k$ and $\langle J \rangle$ has degree $\ell$ when expanded into these variables, the bi-degree of the Segre determinant is $(n/k, n/\ell) = (\ell, k)$. $\qquad \square$

**Remark 7.2.4.** The Segre determinant has further symmetries. It transforms equivariantly under the actions of $(\mathbb{C}^*)^n$ on $\mathcal{B}_{k,n}$ and $\mathcal{B}_{\ell,n}$ obtained by scaling columns of matrix representatives. It is also invariant (up to sign) under permuting the points $A_i \times B_i$ and, when $k = \ell$, exchanging the roles of $A_i$ and $B_i$. The equivariance condition and the degree considerations in Proposition 7.2.2 are enough to deduce, for example, that the polynomial $\mathrm{Seg}_{2,2}$ in Example 7.1.1 is a linear combination of $[12][34]\langle 12\rangle\langle 34\rangle, [13][24]\langle 12\rangle\langle 34\rangle, [12][34]\langle 13\rangle\langle 24\rangle$, and $[13][24]\langle 13\rangle\langle 24\rangle$. The subvector space spanned by invariants of the aforementioned $S_4$ and $S_2$ actions is spanned by $[12][34]\langle 12\rangle\langle 34\rangle + [13][24]\langle 13\rangle\langle 24\rangle$ and $[12][34]\langle 13\rangle\langle 24\rangle - [13][24]\langle 12\rangle\langle 34\rangle$. It follows that the Segre determinant is a linear combination of these.

The brackets in $\mathcal{B}_{k,n}$ satisfy certain relations called the Plücker relations, coming from the relations between maximal minors of a $k \times n$ matrix. As the name implies, these are the same relations defining the Grassmannian in its Plücker embedding. Indeed, $\mathrm{Proj}\mathcal{B}_{k,n}$ equals $\mathrm{Gr}(k, n)$. It is a fact, sometimes called the *Second Fundamental Theorem of Invariant Theory*, that the Plücker relations generate all relations between brackets [Stu93, Theorem 3.1.7]

The bracket algebra comes with a convenient basis in each graded component, which we explain as follows. A *Young tableau* is the filling of a partition diagram with entries in $[n]$, allowing repeats. A *semi-standard Young tableau* has the additional property that the numbers are non-decreasing within each row and strictly increasing within each column. We call a monomial $[I_1] \cdots [I_r]$ in $\mathcal{B}_{k,n}$ *standard* if the $k \times r$ rectangular Young tableau obtained by stacking $I_1, \ldots, I_r$ vertically and then transposing is a semi-standard Young tableau. For example, in $\mathcal{B}_{2,4}$ the monomials $[12][12]$ and $[13][24]$ are standard, but the monomial $[14][23]$ is not. There are 21 degree two monomials in total in $\mathcal{B}_{2,4}$, 20 of which are standard. This use of the term "standard monomial" is consistent with the theory of Gröbner bases: the Plücker relations form a Gröbner basis for the ideal of relations in $\mathcal{B}_{k,n}$ under a certain order called tableau order, and these are the standard monomials [Stu93, Theorem 3.1.7].

The semi-standard monomials of a fixed degree $r$ form a basis for the degree $r$ part of the bracket algebra [Stu93, Corollary 3.1.9]. The *straightening algorithm* applies the Plücker relations to put a polynomial in $\mathcal{B}_{k,n}$ into the unique representation such that every monomial within it is standard. Note that the Segre determinant is an eigenvector of the torus $(\mathbb{C}^*)^{k\ell}$ acting on either $\mathcal{B}_{k,k\ell}$ or $\mathcal{B}_{\ell,k\ell}$. Thus each bracket monomial appearing within it has the same multiset of indices, namely $[n]$. From here on when we speak of the "standard basis" we mean the basis of semi-standard Young tableaux for the multilinear component, in which the multiset of the indices is $[n]$.

**Example 7.2.5.** The Segre determinant $\mathrm{Seg}_{2,4}$ has bi-degree $(4, 2)$ and a total of 22 mono-

mials in the standard basis. Its expansion in standard brackets equals

$$
\begin{aligned}
\mathrm{Seg}_{2,4} \; = \; & - (\langle 1235\rangle\langle 4678\rangle - \langle 1245\rangle\langle 3678\rangle + \langle 1257\rangle\langle 3468\rangle)[13][24][56][78] \\
& - (\langle 1237\rangle\langle 4568\rangle - \langle 1357\rangle\langle 2468\rangle + \langle 1345\rangle\langle 2678\rangle)[12][34][56][78] \\
& - (\langle 1235\rangle\langle 4678\rangle - \langle 1236\rangle\langle 4578\rangle + \langle 1356\rangle\langle 2478\rangle)[12][34][57][68] \\
& + \langle 1234\rangle\langle 5678\rangle[15][26][37][48] + (\langle 1234\rangle\langle 5678\rangle + \langle 1346\rangle\langle 2578\rangle)[12][35][47][68] \\
& - \langle 1347\rangle\langle 2568\rangle[12][35][46][78] - \langle 1345\rangle\langle 2678\rangle[12][36][47][58] \\
& + \langle 1256\rangle\langle 3478\rangle[13][24][57][68] - \langle 1246\rangle\langle 3578\rangle[13][25][47][68] \\
& + \langle 1245\rangle\langle 3678\rangle[13][26][47][58] - \langle 1237\rangle\langle 4568\rangle[14][25][36][78] \\
& + \langle 1236\rangle\langle 4578\rangle[14][25][37][68] - \langle 1235\rangle\langle 4678\rangle[14][26][37][58] \\
& + (\langle 1234\rangle\langle 5678\rangle + \langle 1247\rangle\langle 3568\rangle)[13][25][46][78]. \qquad \diamond
\end{aligned}
\tag{7.9}
$$

One may in principle compute expressions such as Example 7.2.5 by intersecting the expressions in $a_{ij}$ and $b_{ij}$ with the subring of invariants in $\mathbb{C}[a_{ij}] \otimes \mathbb{C}[b_{ij}]$. This intersection is typically computed via elimination algorithms which use Gröbner bases, such as that in [Eis95, Section 15.10.4]. However, these are unlikely to terminate. It is far more efficient to leverage the standard basis and use linear algebra. For instance, to do the above computation we performed the following steps.

1. Do a block Laplace expansion of the Segre matrix (7.1) in the $[I]$ brackets and then straighten to the fourteen standard ones.

2. For each standard monomial $T$ in the $[I]$ brackets:

   i. Define $f_T$ to be the coefficient of $T$ in the variables $b_{ij}$.

   ii. Write $f_T = c_1\langle 1234\rangle\langle 5678\rangle + \ldots + c_{14}\langle 1347\rangle\langle 2568\rangle$, where the $c_i$ are unknown.

   iii. Choose a random degree eight monomial $m$ in the $b_{ij}$ appearing in $f_T$. Its incidence vector with the brackets $\langle J_1\rangle\langle J_2\rangle$ imposes a linear constraint on the $c_i$.

   iv. Repeat until the coefficients $c_i$ are determined.

The final step is linear algebra in a 14-dimensional vector space, so it runs very quickly.

## 7.3   Torus orbits in the Grassmannian

In this section we study torus orbits in the Grassmannian and their Chow–Lam forms; see Chapter 5. The Grassmannian $\mathrm{Gr}(k,n)$ is equipped with an action of $T := (\mathbb{C}^*)^n$. In terms of a matrix parameterization, this may be seen as scaling the columns of a $k \times n$ matrix representative. Given any point $A \in \mathrm{Gr}(k,n)$, we write $\mathcal{T}_A$ for the Zariski closure of the orbit $T \cdot A$ in $\mathrm{Gr}(k,n)$. This is a toric variety of dimension at most $n-1$. It has dimension exactly $n-1$ if $A$ is general, in particular if the Plücker coordinates of $A$ are all non-zero. As in the introduction, we fix positive integers $k$ and $\ell$ and let $n := k\ell$.

**Example 7.3.1** (Torus orbit in $\mathrm{Gr}(2,6)$). Let $k = 2$ and $\ell = 3$, and let $A$ be a generic point in $\mathrm{Gr}(2,6)$ whose Plücker coordinates are non-zero. Then $\mathcal{T}_A$ has dimension $6 - 1 = 5$. This appears as Example 5.3.3, where we computed its Chow–Lam form. We do that computation in more detail now, as it will mirror the proof of Theorem 7.3.2. The Chow–Lam locus lives in $\mathrm{Gr}(3,6)$ and it is parameterized in dual Stiefel coordinates by the matrices

$$
\begin{bmatrix}
a_{11}t_1 & a_{12}t_2 & a_{13}t_3 & a_{14}t_4 & a_{15}t_5 & a_{16}t_6 \\
a_{21}t_1 & a_{22}t_2 & a_{23}t_3 & a_{24}t_4 & a_{25}t_5 & a_{26}t_6 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6
\end{bmatrix} .
$$

Here $(t_1, \ldots, t_6)$ varies over elements of $(\mathbb{C}^*)^6$. However, we could also parameterize it in primal Stiefel coordinates as $3 \times 6$ matrices $B$ such that, for some $t \in (\mathbb{C}^*)^6$, we have

$$
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\
b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\
b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36}
\end{bmatrix}
\cdot \mathrm{diag}(t_1, ..., t_6) \cdot
\begin{bmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22} \\
a_{13} & a_{23} \\
a_{14} & a_{24} \\
a_{15} & a_{25} \\
a_{16} & a_{26}
\end{bmatrix} = 0.
$$

Re-arranging, we obtain the expression

$$
t_1 \begin{bmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{11}b_{31} \\ a_{21}b_{11} \\ a_{21}b_{21} \\ a_{21}b_{31} \end{bmatrix}
+ t_2 \begin{bmatrix} a_{12}b_{12} \\ a_{12}b_{22} \\ a_{12}b_{32} \\ a_{22}b_{12} \\ a_{22}b_{22} \\ a_{22}b_{32} \end{bmatrix}
+ t_3 \begin{bmatrix} a_{13}b_{13} \\ a_{13}b_{23} \\ a_{13}b_{33} \\ a_{23}b_{13} \\ a_{23}b_{23} \\ a_{23}b_{33} \end{bmatrix}
+ t_4 \begin{bmatrix} a_{14}b_{14} \\ a_{14}b_{24} \\ a_{14}b_{34} \\ a_{24}b_{14} \\ a_{24}b_{24} \\ a_{24}b_{34} \end{bmatrix}
+ t_5 \begin{bmatrix} a_{15}b_{15} \\ a_{15}b_{25} \\ a_{15}b_{35} \\ a_{25}b_{15} \\ a_{25}b_{25} \\ a_{25}b_{35} \end{bmatrix}
+ t_6 \begin{bmatrix} a_{16}b_{16} \\ a_{16}b_{26} \\ a_{16}b_{36} \\ a_{26}b_{16} \\ a_{26}b_{26} \\ a_{26}b_{36} \end{bmatrix} = 0.
$$

Thus $B$ is in the Chow–Lam locus of $\mathcal{T}_A$ in $\mathrm{Gr}(3,6)$ if and only if the Segre determinant $\mathrm{Seg}_{2,3}(A, B)$ vanishes. In dual Plücker coordinates on $\mathrm{Gr}(2,6)$ and primal Plücker coordinates on $\mathrm{Gr}(3,6)$, we obtain the expression

$$
\begin{aligned}
\mathrm{Seg}_{2,3} \; = \; & ([12][34][56] + [14][25][36]) \, \langle 123 \rangle \langle 456 \rangle - [13][25][46] \, \langle 124 \rangle \langle 356 \rangle \\
& + [12][35][46] \, \langle 134 \rangle \langle 256 \rangle - [12][34][56] \, \langle 135 \rangle \langle 246 \rangle + [13][24][56] \, \langle 125 \rangle \langle 346 \rangle. \quad \diamond
\end{aligned}
$$

Suppose that $n = k\ell$ for some $\ell \geq 2$. Let $A$ be a general point in $\mathrm{Gr}(k,n)$. Then the Chow–Lam locus is a subvariety of $\mathrm{Gr}(n - \ell, n)$. The analysis in Example 7.3.1 extends to the following result.

**Theorem 7.3.2** (Segre Determinant). *Suppose $k, \ell \geq 2$ and let $n = k\ell$. Fix a point $A$ in $\mathrm{Gr}(k,n)$ with non-zero Plücker coordinates. Then the Chow–Lam form of $\mathcal{T}_A$ in primal Plücker coordinates $B$ on $\mathrm{Gr}(n - \ell, n)$ equals the Segre determinant $\mathrm{Seg}_{k,\ell}(A, B)$.*

Note that Theorem 7.3.2 gives another proof of Proposition 7.2.2. Indeed, by Theorem 5.2.5, the Chow–Lam locus lives in a Grassmannian, and its equation may always be written in the Plücker coordinates of that Grassmannian. We also obtain the result that the Segre determinant is irreducible.

We will need tools from later in this section to prove Theorem 7.3.2 completely. For now, we establish the following lemma.

**Lemma 7.3.3** (Factor of Segre Determinant). *Fix $k, \ell \geq 2$ and let $n = k\ell$. Fix a point $A$ in $\mathrm{Gr}(k, n)$ such that $\dim \mathcal{T}_A = n - 1$. Then the Chow–Lam form of $\mathcal{T}_A$ in primal Plücker coordinates $B$ on $\mathrm{Gr}(n - \ell, n)$ divides the Segre determinant $\mathrm{Seg}_{k,\ell}(A, B)$.*

*Proof.* The Chow–Lam locus is the Zariski closure of the set of points $B$ in $\mathrm{Gr}(n-\ell, n)$ which contain a subspace $t \cdot A$ for some $t \in T$. Representing $B$ in primal Stiefel coordinates and $t \cdot A$ in dual Stiefel coordinates, we are seeking $\ell \times n$ matrices $B$ such that $B \cdot \mathrm{diag}(t_1, \ldots, t_n) \cdot A = 0$. Re-arranging, we obtain the condition that for some $t \in (\mathbb{C}^*)^n$,

$$\sum_{i=1}^{n} t_i (A_i \otimes B_i) = 0. \tag{7.10}$$

If (7.10) holds, then $A_1 \otimes B_1, \ldots, A_n \otimes B_n$ are linearly dependent. Thus the Chow–Lam form is an irreducible factor of the Segre determinant. $\qquad\square$

**Example 7.3.4.** For special matrices $A$, the Segre determinant becomes reducible and the Chow–Lam form is one of the irreducible factors. For instance, let $A$ in $\mathrm{Gr}(2, 4)$ be any point whose Plücker coordinate $[12]$ is zero. Then $\mathcal{T}_A$ is the 3-dimensional variety with ideal generated by $q_{12}$ and the Plücker relation for $\mathrm{Gr}(2, 4)$. By Example 7.1.1,

$$\mathrm{Seg}_{2,2}(A, B) = [13][24]\langle 12 \rangle \langle 34 \rangle.$$

However, the Chow–Lam form of $\mathcal{T}_A$ in primal coordinates is $\langle 34 \rangle$. Indeed, the vanishing of $p_{34} = q_{12}$ exactly cuts out the original variety. The extra factor of $\langle 12 \rangle$ represents the lines passing through the singular point $A' := \mathrm{span}(e_3, e_4)$ in $\mathcal{T}_A$. Indeed, any matrix with $p_{34} = 0$ satisfies $B \cdot A' = 0$. In terms of the proof of Lemma 7.3.3, we have a linear dependence where the coefficients $(0, 0, t_3, t_4)$ are not in $(\mathbb{C}^*)^4$. $\qquad\diamond$

To better predict situations like Example 7.3.4, we will introduce some combinatorial tools to compute the Chow–Lam degree of $\mathcal{T}_A$. The *matroid* of a point $A$ in the Grassmannian $\mathrm{Gr}(k, n)$ is defined by its *bases*, namely the collection of indices $I \in \binom{[n]}{k}$ such that the corresponding Plücker coordinate $q_I$ is nonzero. In general, the variety $\mathcal{T}_A$ will (up to isomorphism) only depend on the underlying matroid of the point $A$; see e.g. [MS21, Proposition 13.12]. Thus we may denote $\mathcal{T}_A$ by $\mathcal{T}_M$, where $M$ is the matroid of $A$. We let $\delta_\lambda(M)$ denote the Schubert coefficient $\delta_\lambda(\mathcal{T}_M)$ and call the collection of these the *Schubert coefficients of the matroid $M$*. In particular, $\alpha(M)$ denotes the Chow–Lam degree of $\mathcal{T}_M$.

The *uniform matroid* $U_{k,n}$ has as its bases all size $k$ subsets of $[n]$. It arises as the matroid of a point whose Plücker coordinates are all nonzero. Klyachko [Kly85] computed the Schubert coefficients of the uniform matroid in terms of dimensions of irreducible $SL_n$-representations. His formula involves a count $\#SSYT(\lambda, n)$ of the number of semi-standard Young tableaux of shape $\lambda$ with entries in $[n]$. This formula comes from representation theory; in that context, the number $SSYT(\lambda, n)$ is the dimension of the irreducible $SL_n$-representation obtained by applying the Schur functor $\mathbb{S}_\lambda$ to the standard representation of $SL_n$. The partition complement $\lambda^c$ is obtained by removing $\lambda$ from a $k \times (n - k)$ rectangle and rotating by 180 degrees.

**Proposition 7.3.5** (Theorem 6 of [Kly85]). *Let $\lambda$ be a partition fitting in a $k \times (n - k)$ rectangle. Then the coefficient $\delta_\lambda(U_{k,n})$ is*

$$\delta_\lambda(U_{k,n}) \;=\; \sum_{i=0}^{k}(-1)^i\binom{n}{i}\#SSYT(\lambda^c, k - i). \tag{7.11}$$

*Proof of Theorem 7.3.2.* By Lemma 7.3.3, the Chow–Lam form divides the Segre determinant. Thus it suffices to show that the Chow–Lam degree of $U_{k,n}$ is $k$. We do this using Klyachko's formula. The complement of $\alpha = (n - r + 1, n - r, \ldots, n - r)$ in the $k \times (n - k)$ rectangle is $\alpha^c = (r - k, \ldots, r - k, r - k - 1)$. Because $\alpha^c$ has $k$ parts, the contribution to the sum in (7.11) is nonzero only when $i = 0$. The semi-standard condition fixes all but the last column of each tableau of shape $\alpha^c$, giving that $\alpha(U_{k,n}) = k$. $\square$

## 7.4   An application to algebraic vision

Computer vision is the study of "cameras," namely linear projections from $\mathbb{P}^3_\mathbb{R}$ to $\mathbb{P}^2_\mathbb{R}$, and how a computer gains information from them. A typical computer vision problem is to reconstruct an object in 3-space from a set of 2-dimensional snapshots. In algebraic vision, the object one is taking a picture of is an algebraic variety in $\mathbb{P}^3_\mathbb{R}$, such as a curve or a collection of points. The survey [KK22] provides an overview of this research area. For the rest of this section we work over the field $\mathbb{R}$.

Fix two configurations $A$ and $B$ of eight ordered points in $\mathbb{P}^2$. One natural question in computer vision is: when are they linear projections of a common configuration in $\mathbb{P}^3$? To answer this, consider the $9 \times 8$ Segre matrix with columns $A_1 \otimes B_1, \ldots, A_8 \otimes B_8$. This matrix has a one-dimensional kernel, provided that the point configurations are sufficiently generic (see [Lon81] for the exact conditions). The kernel of the Segre matrix is a $1 \times 9$ vector. However, it may also be viewed as a $3 \times 3$ matrix, which we denote $F$. Then a necessary and sufficient condition for $A$ and $B$ to have a common recovery is that $F$ has rank two [Lon81]. In algebraic vision $F$ is called the *fundamental matrix*.

If one has instead two configurations of *nine* points, there is an extra condition: namely, the $9 \times 9$ Segre matrix must have a kernel. In Equation (7.12) we express the Segre determinant as a polynomial in brackets $[I]$ and $\langle J \rangle$. This answers a question of Rekha Thomas

about how to write the condition using $\mathrm{SL}_3$-invariants. The expression is relatively sparse, with 110 terms total in the standard basis, which has cardinality $42^2 = 1764$. It is computed in the same manner as $\mathrm{Seg}_{2,4}$, described in the discussion after Example 7.2.5. A short script reproducing the computation may be found at https://github.com/lizziepratt/SegreDeterminant.

$$
\begin{aligned}
\mathrm{Seg}_{3,3} = {} & [123][456][789](3\langle123\rangle\langle457\rangle\langle689\rangle - \langle123\rangle\langle467\rangle\langle589\rangle + 3\langle124\rangle\langle356\rangle\langle789\rangle - 3\langle124\rangle\langle357\rangle\langle689\rangle + \langle124\rangle\langle367\rangle\langle589\rangle - \\
& \langle124\rangle\langle368\rangle\langle579\rangle - \langle125\rangle\langle346\rangle\langle789\rangle + \langle125\rangle\langle347\rangle\langle689\rangle + \langle127\rangle\langle348\rangle\langle569\rangle - \langle134\rangle\langle258\rangle\langle679\rangle - \langle135\rangle\langle247\rangle\langle689\rangle + \\
& \langle145\rangle\langle267\rangle\langle389\rangle + \langle147\rangle\langle258\rangle\langle369\rangle) + [123][457][689](-3\langle123\rangle\langle456\rangle\langle789\rangle + \langle124\rangle\langle368\rangle\langle579\rangle - \langle126\rangle\langle348\rangle\langle579\rangle + \\
& \langle135\rangle\langle246\rangle\langle789\rangle - \langle146\rangle\langle258\rangle\langle379\rangle) + [123][458][679](-\langle124\rangle\langle367\rangle\langle589\rangle - \langle125\rangle\langle346\rangle\langle789\rangle + \langle126\rangle\langle347\rangle\langle589\rangle + \\
& \langle146\rangle\langle257\rangle\langle389\rangle) + [123][467][589](\langle123\rangle\langle456\rangle\langle789\rangle - \langle124\rangle\langle358\rangle\langle679\rangle + \langle125\rangle\langle348\rangle\langle679\rangle + \langle134\rangle\langle256\rangle\langle789\rangle - \\
& \langle135\rangle\langle246\rangle\langle789\rangle + \langle145\rangle\langle268\rangle\langle379\rangle) + [124][356][789](-3\langle123\rangle\langle456\rangle\langle789\rangle + \langle123\rangle\langle468\rangle\langle579\rangle + \langle135\rangle\langle247\rangle\langle689\rangle - \\
& \langle135\rangle\langle267\rangle\langle489\rangle - \langle137\rangle\langle258\rangle\langle469\rangle) + [124][357][689](3\langle123\rangle\langle456\rangle\langle789\rangle - \langle123\rangle\langle468\rangle\langle579\rangle - \langle135\rangle\langle246\rangle\langle789\rangle + \\
& \langle136\rangle\langle258\rangle\langle479\rangle) + [124][358][679](\langle123\rangle\langle467\rangle\langle589\rangle - \langle136\rangle\langle257\rangle\langle489\rangle) + [124][367][589](-\langle123\rangle\langle456\rangle\langle789\rangle + \\
& \langle123\rangle\langle458\rangle\langle679\rangle + \langle135\rangle\langle246\rangle\langle789\rangle - \langle135\rangle\langle268\rangle\langle479\rangle) + [124][368][579](\langle123\rangle\langle456\rangle\langle789\rangle - \langle123\rangle\langle457\rangle\langle689\rangle + \\
& \langle135\rangle\langle267\rangle\langle489\rangle) + [125][346][789](\langle123\rangle\langle456\rangle\langle789\rangle + \langle123\rangle\langle458\rangle\langle679\rangle - \langle123\rangle\langle468\rangle\langle579\rangle - \langle134\rangle\langle257\rangle\langle689\rangle + \\
& \langle134\rangle\langle267\rangle\langle589\rangle + \langle137\rangle\langle248\rangle\langle569\rangle) + [125][347][689](-\langle123\rangle\langle456\rangle\langle789\rangle + \langle123\rangle\langle468\rangle\langle579\rangle + \langle134\rangle\langle256\rangle\langle789\rangle - \\
& \langle136\rangle\langle248\rangle\langle579\rangle) + [125][348][679](-\langle123\rangle\langle467\rangle\langle589\rangle + \langle136\rangle\langle247\rangle\langle589\rangle) + [125][367][489](-\langle134\rangle\langle256\rangle\langle789\rangle + \\
& \langle134\rangle\langle268\rangle\langle579\rangle) - [125][368][479]\langle134\rangle\langle267\rangle\langle589\rangle + [126][347][589](-\langle123\rangle\langle458\rangle\langle679\rangle + \langle135\rangle\langle248\rangle\langle679\rangle) + \\
& [126][348][579](\langle123\rangle\langle457\rangle\langle689\rangle - \langle135\rangle\langle247\rangle\langle689\rangle) - [126][357][489]\langle134\rangle\langle258\rangle\langle679\rangle + \\
& [126][358][479]\langle134\rangle\langle257\rangle\langle689\rangle + [127][348][569](-\langle123\rangle\langle456\rangle\langle789\rangle + \langle135\rangle\langle246\rangle\langle789\rangle) - \\
& [127][358][469]\langle134\rangle\langle256\rangle\langle789\rangle + [134][256][789](-\langle123\rangle\langle467\rangle\langle589\rangle - \langle125\rangle\langle347\rangle\langle689\rangle + \langle125\rangle\langle367\rangle\langle489\rangle + \\
& \langle127\rangle\langle358\rangle\langle469\rangle) + [134][257][689](\langle125\rangle\langle346\rangle\langle789\rangle - \langle126\rangle\langle358\rangle\langle479\rangle) + [134][258][679](\langle123\rangle\langle456\rangle\langle789\rangle + \\
& \langle126\rangle\langle357\rangle\langle489\rangle) + [134][267][589](-\langle125\rangle\langle346\rangle\langle789\rangle + \langle125\rangle\langle368\rangle\langle479\rangle) - [134][268][579]\langle125\rangle\langle367\rangle\langle489\rangle + \\
& [135][246][789](-\langle123\rangle\langle457\rangle\langle689\rangle + \langle123\rangle\langle467\rangle\langle589\rangle + \langle124\rangle\langle357\rangle\langle689\rangle - \langle124\rangle\langle367\rangle\langle589\rangle - \langle127\rangle\langle348\rangle\langle569\rangle) + \\
& [135][247][689](\langle123\rangle\langle456\rangle\langle789\rangle - \langle124\rangle\langle356\rangle\langle789\rangle + \langle126\rangle\langle348\rangle\langle579\rangle) - [135][248][679]\langle126\rangle\langle347\rangle\langle589\rangle + \\
& [135][267][489](\langle124\rangle\langle356\rangle\langle789\rangle - \langle124\rangle\langle368\rangle\langle579\rangle) + [135][268][479]\langle124\rangle\langle367\rangle\langle589\rangle - \\
& [136][247][589]\langle125\rangle\langle348\rangle\langle679\rangle + [136][248][579]\langle125\rangle\langle347\rangle\langle689\rangle + [136][257][489]\langle124\rangle\langle358\rangle\langle679\rangle - \\
& [136][258][479]\langle124\rangle\langle357\rangle\langle689\rangle - [137][248][569]\langle125\rangle\langle346\rangle\langle789\rangle + [137][258][469]\langle124\rangle\langle356\rangle\langle789\rangle + \\
& [145][267][389](-\langle123\rangle\langle456\rangle\langle789\rangle + \langle123\rangle\langle468\rangle\langle579\rangle) - [145][268][379]\langle123\rangle\langle467\rangle\langle589\rangle - \\
& [146][257][389]\langle123\rangle\langle458\rangle\langle679\rangle + [146][258][379]\langle123\rangle\langle457\rangle\langle689\rangle - [147][258][369]\langle123\rangle\langle456\rangle\langle789\rangle.
\end{aligned}
$$

$$(7.12)$$

**Remark 7.4.1.** In principle, one may also compute the rank condition on $F$ in terms of brackets. However, the degree in brackets gets quite large. For eight points in $\mathbb{P}^2$, by Cramer's rule each entry of $F$ is a $8 \times 8$ minor of the Segre matrix of the eight points. Thus one may express the rank condition as a polynomial of bi-degree $(8, 8)$ in the brackets $[I]$ and $\langle J \rangle$ on two copies of $\mathcal{B}_{3,8}$.

We close this section with a theorem which states that in general, existence of a common recovery implies that the fundamental matrix has rank at most two. When there are $k^2$ or more points, there is an additional condition that the Segre matrix must drop rank. Theorem

7.4.2 is a consequence of the more general Theorem 2 in [BGT17] (thanks to Timothy Duff for pointing this out). However, we include a self-contained proof for our setting.

**Theorem 7.4.2.** *Let $A$ and $B$ be distinct configurations of $k^2 - 1$ ordered points in $\mathbb{P}^{k-1}$. Suppose that there exists a configuration of ordered points $C$ in $\mathbb{P}^{2k-3}$ and linear projections $\pi_1, \pi_2 : \mathbb{P}^{2k-3} \dashrightarrow \mathbb{P}^{k-1}$ such that $\pi_1(C) = A$ and $\pi_2(C) = B$. Furthermore, suppose that the points are sufficiently generic for the kernel of the $k^2 \times (k^2 - 1)$ Segre matrix to be one-dimensional. Then the kernel $F$, when viewed as a $k \times k$ matrix, has rank at most two.*

*Proof.* Our strategy is to construct a matrix $F$ of rank at most two, such that for each pair $(a, b)$ of points in $\mathbb{P}^{k-1}$ which are projections of a common point in $\mathbb{P}^{2k-3}$, the product $b^T F a$ vanishes. Since $b^T F a = \sum_{i,j} F_{ij} a_i b_j$, this would imply that the $1 \times k^2$ flattening of $F$ is in the left kernel of the $k^2 \times (k^2 - 1)$ Segre matrix. We first fix a partial inverse $\varphi$ to $\pi_1$. Define $V := \widehat{\pi_2(\ker \pi_1)}$, which is a $(k-2)$-dimensional subspace of $\mathbb{R}^k$. Let $M$ be the map $\pi_2 \circ \varphi$. In summary, we have the data in the following commutative diagram:



Then consider the map

$$\tilde{F} : \mathrm{Gr}(1, k) \dashrightarrow \mathrm{Gr}(k - 1, k)$$
$$x \mapsto V \oplus M(x).$$

This map is linear in $x$ in the sense that it is component-wise given by linear forms. More precisely, representing $V$ with a $(k-2) \times k$ matrix and $M(x)$ with a $1 \times k$ matrix of the same names, the dual Plücker coordinates of $\tilde{F}(x)$ are the $(k-1)$-minors of the $(k-1) \times k$ matrix

$$\left[ \begin{array}{c} M(x) \\ \hline V \end{array} \right] \tag{7.13}$$

The map $\tilde{F}$ is thus represented by a matrix, which we will call $F$. Its base locus is $\mathbb{P}(V)$. Thus the rank of $F$ is two and the image is all hyperplanes in $\mathbb{P}^{k-1}$ containing $\mathbb{P}(V) = \mathbb{P}(\ker F)$. Now, suppose that there exists $c$ such that $\pi_1(c) = a$ and $\pi_2(c) = b$. Then $c$ is in the projectivization of the vector space $\varphi(a) \oplus \ker \pi_1$. So $\pi_2(c) = b$ is on the line $F(a)$. In terms of matrix representatives, we then have $b^T F a = 0$. $\square$

**Remark 7.4.3.** Theorem 7.4.2 gives a necessary condition for two point configurations to be projections of a common configuration. Theorem 2 of [BGT17] proves this is also sufficient, and Section 5.1.2 gives an algorithm for reconstructing the points and projection matrices.

# 7.5 The Segre coefficient variety

In this section, we introduce the *Segre coefficient variety*, which parameterizes Segre determinants of points in $\mathrm{Gr}(k, k\ell)$. The ambient space of the Segre coefficient variety is the projectivization of the vector space spanned by degree $k$ monomials in the $B$-variables. We prove in Theorem 7.5.2 that the linear span of these Segre determinants is as large as possible; namely that it equals the *multilinear component*. This results in Corollary 7.5.3, which states that for $k = 2$, the Segre coefficient variety recovers the GIT quotient $\mathcal{M}_{0,2\ell}$ parameterizing configurations of labelled points on the projective line.

Let $\mathrm{Gr}(k, k\ell)^\circ \subset \mathrm{Gr}(k, k\ell)$ be the Zariski open subset of points whose matroid is uniform. Consider the map

$$\pi : \mathrm{Gr}(k, k\ell)^\circ \to \mathbb{P}H^0(\mathrm{Gr}(\ell, k\ell), \mathcal{O}_{\mathrm{Gr}(\ell, k\ell)}(k)) \tag{7.14}$$
$$A \mapsto \mathrm{Seg}(A, B).$$

The map $\pi$ sends a point $A$ to $\mathrm{Seg}(A, B)$, viewed as a polynomial in the $B$-coordinates. While $\pi$ is defined without choosing a basis for the target space, it is often convenient to take a basis of standard $B$-monomials of degree $k$ for $\mathcal{O}_{\mathrm{Gr}(\ell, k\ell)}(k)$, as in Example 7.5.1. Then $\pi$ sends $A$ to the vector of $A$-coefficients of the standard $B$-monomials appearing in $\mathrm{Seg}_{k,\ell}(A, B)$. We define the *Segre coefficient variety* $\mathrm{Coeff}(\mathrm{Seg}_{k,\ell})$ as the Zariski closure of the image of $\pi$.

**Example 7.5.1** (Segre cubic). From Example 7.3.1 we get the map

$$\pi : \mathrm{Gr}(2, 6)^\circ \to \mathbb{P}^4$$
$$A \mapsto ([12][34][56] + [14][25][36], \tag{7.15}$$
$$- [13][25][46], \ [12][35][46], \ -[12][34][56], \ [13][24][56]).$$

Its image is cut out by the following degree 3 polynomial, which is known as the *Segre cubic*:

$$x_0 x_1 x_3 - x_1 x_2 x_3 - x_0 x_2 x_4 - x_1 x_2 x_4 - x_1 x_3 x_4 - x_2 x_3 x_4.$$

The corresponding variety is the *Segre cubic threefold*. It is the unique (up to isomorphism) cubic hypersurface in $\mathbb{P}^4$ with ten ordinary double points, the maximum possible [Kal86]. ◇

The Segre coefficient variety recovers a known construction for $k = 2$, namely the GIT quotient $(\mathbb{P}^1)^{2\ell} // \mathrm{SL}_2$ parameterizing configurations of $2\ell$ distinct points on a projective line. This is defined as $\mathrm{Proj}\, R$, where $R$ is the graded ring of $SL_2$-invariants of $2\ell$ ordered points in $\mathbb{P}^1$. This ring is studied by Howard, Millson, Snowden, and Vakil in [How+12], and they give generators for the ideal of relations between these invariants. The equivalence of that construction with our construction rests on Theorem 7.5.2.

**Theorem 7.5.2.** *Every monomial of the form $[I_1] \cdots [I_\ell]$ with $I_1 \cup \ldots \cup I_\ell = [k\ell]$ appears in the linear span of the coefficients of the $B$-monomials in $Seg_{k,\ell}(A, B)$.*

*Proof.* Let $V$ be the vector space spanned by monomials $[I_1] \cdots [I_\ell]$ such that $I_1 \cup \ldots \cup I_\ell = [k\ell]$. Suppose for contradiction that the coefficients of the Segre determinant do not span $V$. Then there is a linear relation among them when $\mathrm{Seg}_{k,\ell}(A, B)$ is written in the basis of standard monomials. Thus the Segre coefficient variety lies in a hyperplane in $\mathbb{P}(V)$.

We show that the Segre coefficient variety does not lie in a hyperplane, so that the assumption is false. To do this, we will produce a collection of points in $\mathrm{Coeff}(\mathrm{Seg}_{k,\ell})$ whose linear span is all of $\mathbb{P}(V)$.

Consider the point $A$ in $\mathrm{Gr}(k, k\ell)$ given by the rowspan of a matrix where columns $(i-1)\ell + 1$ through $i\ell$ are equal to $e_i$, for $1 \le i \le k$. The Segre matrix in (7.1) develops a block form in this case. Explicitly, if we choose the basis $e_1 \otimes e_1, e_1 \otimes e_2, \ldots, e_1 \otimes e_\ell, \ldots, e_k \otimes e_\ell$ for $V \otimes W$, then the Segre matrix will have $k$ blocks of size $\ell \times \ell$ along its diagonal and be zero elsewhere:

$$\mathrm{Seg}_{k,\ell}(A, B) = \det \begin{bmatrix} b_{11} & \ldots & b_{1\ell} & & & & \\ \vdots & \ddots & \vdots & & 0 & & \\ b_{\ell 1} & \ldots & b_{\ell,\ell} & & & & \\ & & & \ddots & & & \\ & 0 & & & b_{1,(k-1)\ell+1} & \ldots & b_{1,k\ell} \\ & & & & \vdots & \ddots & \vdots \\ & & & & b_{\ell,(k-1)\ell+1} & \ldots & b_{\ell,k\ell} \end{bmatrix}.$$

The determinant (up to scalar) factors as $\langle 1 \cdots \ell \rangle \langle \ell + 1 \cdots 2\ell \rangle \cdots \langle (k-1)\ell + 1 \cdots k\ell \rangle$, where as before the angle brackets are minors of the $B$-matrix. Thus the Segre determinant of $\mathcal{T}_A$ is nonzero, so the map $\pi$ extends from $\mathrm{Gr}(k, k\ell)^\circ$ to $A$. By symmetry, we may take any point $A$ given as the rowspan of a matrix whose columns are partitioned into $k$ classes of $\ell$ pairwise parallel elements. By continuity, the image $\pi(A)$ will be in the closure of the Segre coefficient variety. The collection of these points together spans all of $\mathbb{P}(V)$. $\qquad \square$

We call the span of the monomials in Theorem 7.5.2 the *multilinear component* of the vector space of degree $k$ monomials in the Plücker coordinates of $\mathrm{Gr}(k, k\ell)$. The projectivization of the latter vector space appears as the target of the map (7.14). In this language, Theorem 7.5.2 says that the monomials appearing as coefficients of Segre determinants of points in $\mathrm{Gr}(k, k\ell)$ span the whole multilinear component.

In the case $k = 2$, another spanning set of the multilinear component is given by Kempe's *matchings*. These are directed planar graphs with $n$ labelled vertices embedded in a disc, where vertices are in clockwise order and each vertex has degree one; see Figure 7.1.

To each such graph $\Gamma$ we associate the $\mathrm{SL}_2$-invariant

$$X_\Gamma := \prod_{(i \to j) \in \Gamma} (x_i y_j - x_j y_i).$$

Kempe proved that the set of $X_\Gamma$ span the space of invariants of $\mathrm{SL}_2$ acting on $V^{\otimes n}$, where $V$ is 2-dimensional and $\mathrm{SL}_2$ takes the standard action on each component of the tensor product.

Figure 7.1: Two Kempe diagrams for $n = 6$

Furthermore, the non-crossing matchings, i.e. the graphs $\Gamma$ with no internal crossing edges, form a basis [Kem94]. Denoting the multilinear component by $W$, the map

$$\psi : (V^{\otimes n})^{\mathrm{SL}_2} \to W$$
$$X_\Gamma \mapsto \prod_{(i \to j) \in \Gamma} [ij] \tag{7.16}$$

is an isomorphism of vector spaces. That is, consider the expansion of each Plücker monomial in the $2 \times 2$ minors of a $2 \times n$ matrix with entries $x_{11}, \ldots, x_{2n}$. Then a set of Plücker monomials is dependent if and only if the expansions are dependent in $\mathbb{C}[x_{11}, \ldots, x_{2n}]$. This identification of vector spaces leads to the following corollary.

**Corollary 7.5.3.** *The variety* $\mathrm{Coeff}(Seg_{2,\ell})$ *is isomorphic to the GIT quotient* $(\mathbb{P}^1)^{2\ell} /\!/_w \mathrm{SL}_2$, *where* $w = 1^{2\ell}$ *is the linearization.*

*Proof.* Let $n := 2\ell$. Let $R$ denote the ring of invariants of $\mathrm{SL}_2$ acting on $(\mathbb{P}^1)^n$, where we choose the linearization $(1, \ldots, 1)$ of the action. That is,

$$R := \bigoplus_{d=0}^{\infty} H^0((\mathbb{P}^1)^n, \mathcal{O}(d, \ldots, d)).$$

By Kempe's theorem, $R$ is generated in degree one [Kem94]; for a more modern proof see [How+12, Theorem 2.1]. Let $R^{(1)}$ denote the degree one piece of $R$. The quotient map $\mathrm{Sym}(R^{(1)}) \to R$ gives a map $\mathrm{Proj}\, R \hookrightarrow \mathrm{Proj}\,\mathrm{Sym}(R^{(1)})$. Thus we obtain an embedding of the GIT quotient $\mathrm{Proj}\, R$ into the projectivization $\mathbb{P}(R^{(1)})$ of the vector space $R^{(1)}$. By $\varphi$ we denote the composition

$$\varphi : (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2 \hookrightarrow \mathbb{P}(R^{(1)}).$$

The stable locus of $\varphi$ contains the dense open subset $U$ of tuples of $n$ distinct points in $(\mathbb{P}^1)^n$; see e.g. [DO88, Section 7.1]. Because $U$ is dense and $\varphi$ is continuous, we have that the Zariski closure $\overline{\varphi(U)}$ is equal to the Zariski closure of the image of $\varphi$.

Let $\mathrm{Mat}(2,n)^\circ$ denote $2 \times n$ matrices whose maximal minors are nonzero. By composing with the projection map from $\mathrm{Mat}(2,n)^\circ$ to $U$, we obtain

$$G : \mathrm{Mat}(2,n)^\circ \to \mathbb{P}(R^{(1)}).$$

This map is invariant with respect to the action of $\mathrm{SL}_2$ on $\mathrm{Mat}(2,n)^\circ$ by left multiplication. Thus it factors through the Grassmannian $\mathrm{Gr}(2,n)$, and in particular through the open subset $\mathrm{Gr}(2,n)^\circ$ of points whose Plücker coordinates are nonzero. Let $H : \mathrm{Gr}(2,n)^\circ \to \mathbb{P}(R^{(1)})$ denote the resulting map, of which the closure of the image is the embedded GIT quotient.

We will show that the closure of the image of $H$ is linearly equivalent to the image of our map $\pi$ defining the Segre coefficient variety, which will prove the theorem. Let $\Gamma_1, \ldots, \Gamma_r$ be the non-crossing matchings, whose corresponding invariants span $R^{(1)}$. By composing with the isomorphism $\psi$ from (7.16), we may write $H$ in our graphical basis as

$$\mathrm{Gr}(2,n)^\circ \to \mathbb{P}(W)$$
$$A \mapsto [\ \prod_{(i \to j) \in \Gamma_1} [ij] : \ldots : \prod_{(i \to j) \in \Gamma_r} [ij]]$$

where $[ij]$ denote the Plücker coordinates of $A$. These are certain monomials $[I_1] \cdots [I_\ell]$ where $I_1 \cup \ldots \cup I_\ell = [2\ell]$. Theorem 7.5.2 tells us that every such monomial lies in the linear span of the coefficients of the Segre polynomial, so the image is linearly isomorphic to the Segre coefficient variety. $\qquad\square$

If we drop the assumption that $I_1 \cup \ldots \cup I_\ell = [k\ell]$ and instead range over all sets $\{I_1, \ldots, I_\ell\}$ with entries in $[k\ell]$, the collection of monomials $[I_1] \cdots [I_\ell]$ uniquely determines the torus orbit closure; see e.g. [MS21, Chapter 13]. However, the map $\pi$ is not injective on generic torus orbit closures in general. This is because Kempe's theorem is not true in general for $k \geq 3$.

**Example 7.5.4** (Six points in $\mathbb{P}^2$). From Example 7.3.1 we have that the Segre coefficient map is linearly equivalent to the map

$$\pi' : \mathrm{Gr}(3,6)^\circ \to \mathbb{P}^4$$
$$A \mapsto ([123][456], [124][356], [125][346], [134][256], [135][246]). \tag{7.17}$$

Note that the dimension of the Chow quotient $\mathrm{Gr}(3,6) \,!\, /\mathbb{C}^6$ is four; indeed, if we interpret this Chow quotient as a parameter space of points in $\mathbb{P}^2$, then we may computing this dimension by fixing the first four points, and leaving two points free. Thus the map (7.17) is surjective. We will argue that it is a $2:1$ cover.

Let $R$ be the ring of invariants of six points in space, namely

$$R = (\bigoplus_d \mathrm{Sym}^d(\mathbb{C}^3) \otimes \cdots \otimes \mathrm{Sym}^d(\mathbb{C}^3))^{\mathrm{SL}(3)}, \tag{7.18}$$

where there are six factors in the tensor product. The dimension of the $d$graded piece of $R$ is equal to the number of semi-standard Young tableaux with shape $(2d, 2d, 2d)$ and filling $(d, d, d, d, d, d)$. The dimensions of the first few graded components are $1, 5, 16, 40, 85, 161, \ldots$. We determine the Hilbert polynomial of $R$ by finding the dimensions of the graded pieces from degree 5 to degree 9 and interpolating. This gives

$$h(x) = (1/12)x^4 + (1/2)x^3 + (17/12)x^2 + 2x + 1.$$

The degree one part of the ring is spanned by the brackets

$$x_0 = [123][456], \ x_1 = [124][356], \ x_2 = [125][346], \ x_3 = [134][256], \ x_4 = [135][246].$$

We claim that this ring is generated by its degree one piece and a single degree two generator $y := [123][145][246][356] - [124][135][236][456]$. There is a single relation in degree four, where $x_0$ through $x_4$ correspond to standard monomials:

$$
\begin{aligned}
x_0 x_1^2 x_4 + x_0 x_1 x_4^2 - (x_0^2 x_1 x_4 + x_0 x_1 x_2 x_4 + x_0 x_1 x_3 x_4) + \\
((x_0^2 + x_0 x_2 + x_0 x_3 + x_2 x_3) - (x_0 x_1 + x_0 x_4 + x_1 x_4))y + y^2 = 0.
\end{aligned}
\tag{7.19}
$$

Thus the ring of invariants $R = S[y]$ is an integral extension of $S$ of degree two. One could also deduce that $S \neq R$ purely from the Hilbert function of $R$; indeed, the dimension of $R^{(2)}$ is 16, but the space of degree two polynomials in five variables is only 15-dimensional. The Hilbert series of the ring is

$$h(t) = \frac{1 - t^4}{(1 - t)^5(1 - t^2)} = \frac{1 + t^2}{(1 - t)^5}.$$

This example can also be found in [DO88]. The quartic polynomial $y$ is zero if and only if the six points lie on a conic.

For a concrete example of two points in the source of (7.18) which map to the same point in $\mathbb{P}^4$, we parameterize the Grassmannian by $3 \times 6$ matrices. Then by taking products of $3 \times 3$ minors as in (7.17) the following two matrices have nonzero Plücker coordinates and both map to the point $(-2, 2, 9, 2, 8)$ under the map $\pi'$:

$$
p = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 & 5 \end{bmatrix}, \qquad
q := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{bmatrix}.
$$

However, their torus orbit closures are different. We may see this by noting that $[123][145][246][356]$ evaluates to $-8$ for the first point and $4$ for the second in the affine chart given by $[123] = 1$, and is constant on torus orbits. $\diamond$

Recall that by Theorem 7.3.2, the Segre determinant computes the Chow–Lam form of a torus orbit closure. From the point of view of the Chow–Lam form, this gives us a new family of varieties which are distinct but have the same Chow–Lam form, thus building on the work of Chapter 6.

## 7.6 Questions and Extensions

The general case of $k = 3$ would be interesting to explore. In this case, when $n$ is a multiple of three, the ring of invariants $R$ is defined as in (7.18), but with $n$ factors in the tensor product. In the case where $n$ is not a multiple of three, we use the grading

$$R_n = \bigoplus_d \operatorname{Sym}^{3d}(\mathbb{C}^3) \otimes \cdots \otimes \operatorname{Sym}^{3d}(\mathbb{C}^3). \tag{7.20}$$

where again we have $n$ factors in the tensor product.

Let $S$ be the subalgebra of $R$ generated by the degree one part. The inclusion $S \subset R$ is often proper, including in the case $n = 6$ and 7. The degree $[k(S) : k(R)]$ is equal to the number of general torus orbits in $\operatorname{Gr}(3, n)$ which share a given Chow–Lam form. Thus it is a meaningful number to compute from the point of view of Chow–Lam recovery.

**Question 7.6.1.** Let $R$ be the invariant ring of $n$ points in the plane, as in (7.18) for $n$ a multiple of three, or (7.20) otherwise. Let $S$ be the sub-algebra generated by the degree one part of this ring. Is the extension $k(R)/k(S)$ finite, and if so what is its degree?

**Example 7.6.2** (Seven points in the plane). The Hilbert polynomial of $R_7$ is

$$(413/24)x^6 + (413/8)x^5 + (1673/24)x^4 + (427/8)x^3 + (301/12)x^2 + 7x + 1.$$

The first few values of the Hilbert function are

$$1, 225, 4411, 32425, 145041, 481111, 1307125, 3079161, 6517225, 12691981, 23123871, \ldots$$

and the Hilbert series is

$$h(t) = (t^6 + 218t^5 + 2857t^4 + 6238t^3 + 2857t^2 + 218t + 1)/(1 - t)^7.$$

The configuration space of seven points in $\mathbb{P}^2$ has dimension six, so the graded coordinate ring has Krull dimension seven. We computed using Julia and numerical methods that a 3786-dimensional subspace of the degree two elements are in the sub-algebra generated by the degree one part of the ring. This leaves a 625-dimensional vector space in the degree two part which is not generated by the degree one part. It would be interesting to interpret these extra generators geometrically. ◇

We also remark that there are bases other than the bracket basis for the space of $\operatorname{SL}(k)$-invariants of a tensor product $\mathbb{C}^k \otimes \ldots \otimes \mathbb{C}^k$ of $\operatorname{SL}(k)$-representations. A spanning set is given by tensor diagrams for general $\operatorname{SL}(k)$. For the special case of $k = 2$, we recover Kempe's diagrams, which were featured in Section 7.5. For the special case of $\operatorname{SL}(3)$, these invariants are also described by Kuperberg's webs. There is a rotation-invariant subset of these called the *non-elliptic webs*, which form bases for each graded piece of the ring of invariants.

The difficulty with the web basis is that it does not behave well under multiplication for $k$ greater than two. Given a web, there is no easy way to tell whether it factors into the

product of other webs; furthermore, given two non-elliptic webs, their product is typically not non-elliptic. Finally, it is more difficult (in the author's opinion) to write down the SL(3)-invariant corresponding to each web as a polynomial in the matrix entries of a $3 \times n$ matrix. It requires a signed sum similar to the definition of the determinant. Still, there is a nice interpretation of Example 7.5.4 in terms of webs, which we explain as follows.

**Example 7.6.3.** Consider six points in $\mathbb{P}^2$ and draw a hexagon connecting them in any order. By Bézout's theorem, each pair of opposite edges of the hexagon meets in a single point. By Pascal's theorem, these six points lie on a conic if and only those three points lie on a line. Translating this into the language of webs, we find the web in Figure 7.3. The
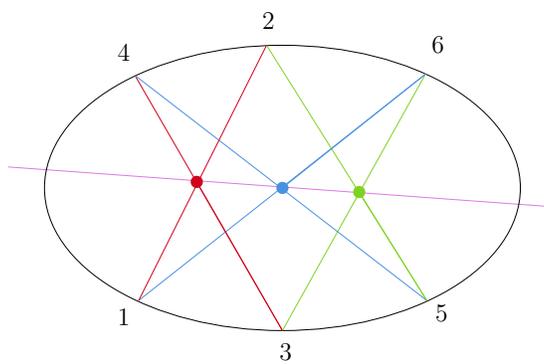


Figure 7.2: Visual interpretation of Pascal's theorem

argument in Example 7.5.4 tells us that there is no way to write the web invariant in Figure 7.3 as a sum of products of degree one web invariants. ◇
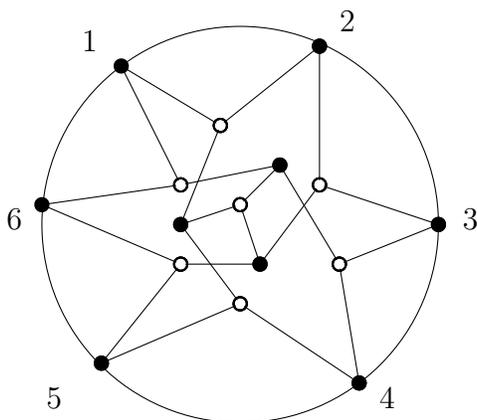


Figure 7.3: The web corresponding to the condition for 6 points to be co-conic

Part II of this thesis has been about generalizations of classical discriminants which appear in particle physics. The Chow–Lam form is a particular discriminant arise naturally from projections between Grassmannians, which of intrinsic interest to algebraic geometry, as well as to the study of the amplituhedron. There are many other discriminants that appear in the study of scattering amplitudes and Feynman integrals, including the $A$-discriminant in Landau analysis [24] and a new family of discriminants capturing leading singularities of Feynman integrals [**TODO papers appearing next week**]. Their degrees are important invariants in the study of amplitudes. We expect that collaborations with the physics community will continue to be fruitful for both physics and mathematics, and look forward to further developing these connections.

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# Appendix A

# Poset of bases of $W_{2,2,n}$

This section contains supporting computations for the proof of Theorem 3.3.10. There are 47 combinatorial types of bases in the matroid $W_{2,2,n}$. They form a poset under cutting and gluing as discussed in Subsection 3.3.4. This poset is pictured in Figure A.1. The bases are pictured, in the order of the Hasse diagram, in Figure A.2. The graphs at the top of the poset are $K_4$ (number 1), the house-shaped graph (number 3), and the cycle (number 12).
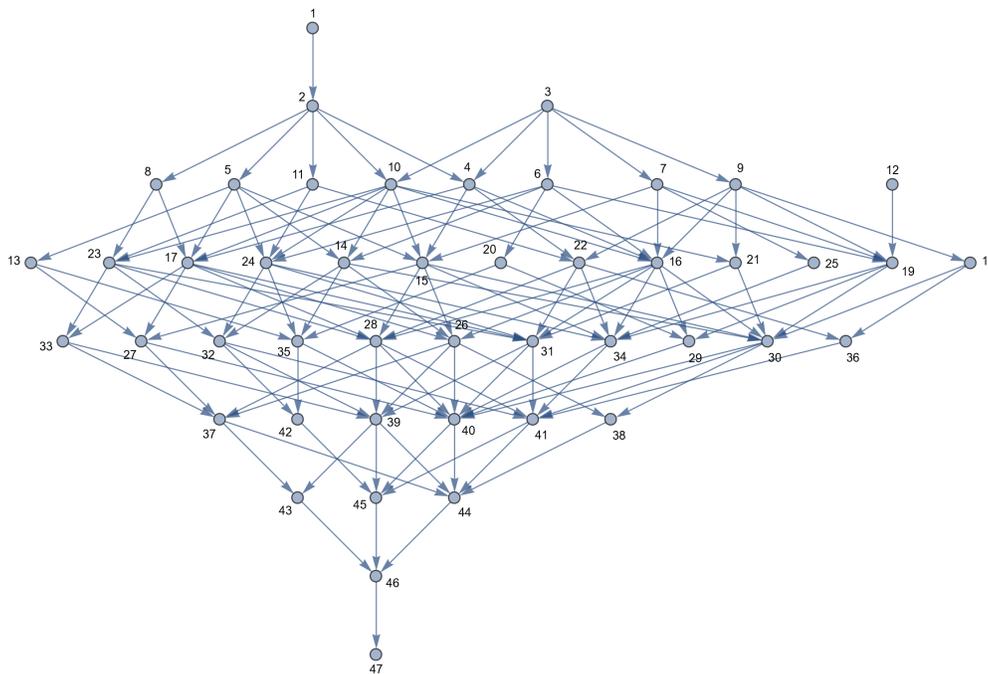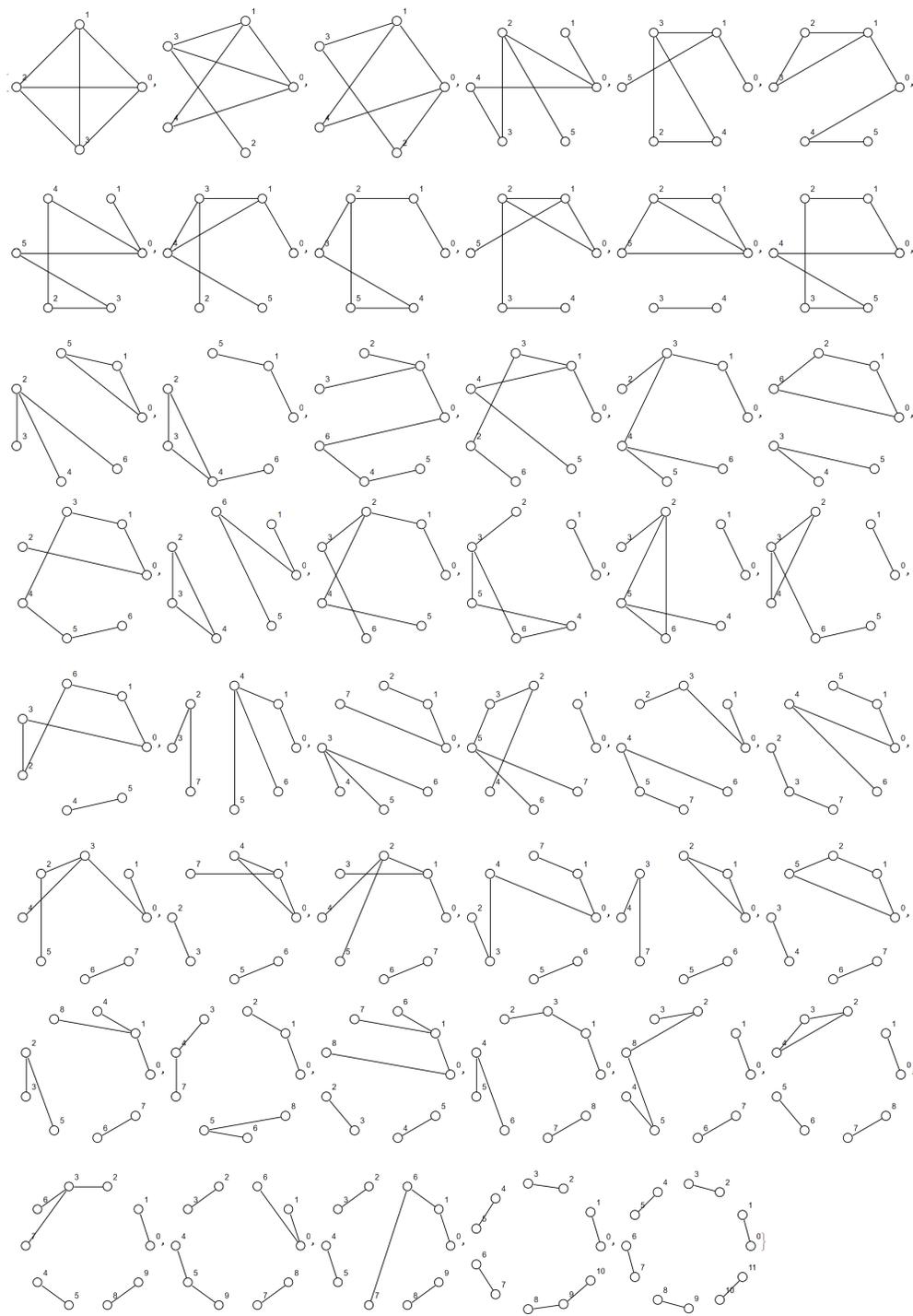


Figure A.1: The poset of bases of $W_{2,2,n}$ under cutting

Figure A.2: Bases of $W_{2,2,n}$