Multigraded Hurwitz Forms

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MaPhyAG seminar

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Discriminant: a polynomial in the coefficients of some polynomial system telling you when it has an unexpected # of solutions.

eg: $\#V(\underline{a}x^2 + \underline{b}x + \underline{c}) < 2 \iff b^2 - 4ac = 0.$ (if $a \neq 0$)

(?)

$\#V(ax^3 + bx^2 + cx^2 + d) < 3 \iff b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = 0.$

(Cardano 1500s)

$$\#V\begin{pmatrix} a_0 + a_1x + a_2x^2 + a_3y + a_4xy \\ b_0 + b_1x + b_2x^2 + b_3y + b_4xy \end{pmatrix} \underset{=}{<} 3 \iff \text{degree } (4,4) \text{ polynomial in } a_i, b_i \text{ vanishes}$$

- 2 variables
- no y^2

(Gelfand, Kapranov, Zelevinsky '94 / Sturmfels '17)

Discriminants in the Sciences

① Nash discriminant in algebraic game theory

[Abo-Portokal-Sodomaco '26]

- n players, d strategies, n payoff tables with d^n entries
- for which payoffs does the # of equilibria drop?

② Leading singularity discriminant for Feynman integrals in particle physics

[Hollering-Mazzucchelli-Parisi-Sturmfels '26]

see also [Fevola-Mizera-Telen '24]

- integrals depend on parameters $\int \frac{dx}{(x-a)(x-b)}$

~~multigraded~~ Hurwitz forms

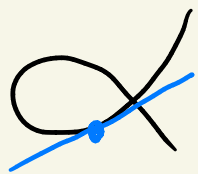
- for which parameters does the integral develop poles or branch cuts?

Setup: $X \subseteq \mathbb{P}^n$ codim d. Consider

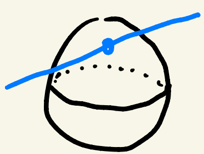
$$\overline{\{(p, L) : \dim L = d,\ L \text{ is tangent to } X \text{ at } p \in X_{reg}\}} \subseteq X \times G(d,n)$$

The Hurwitz locus $\mathcal{H}_X$ is the projection to $G(d,n)$.

Its equation is the Hurwitz form H_X. (if $\mathcal{H}_X$ is a hypersurface;
else $H_X := 1$)



$\subseteq \mathbb{P}^2$ $\quad\quad$ $\mathcal{H}_X \subseteq G(1,2)$



$\subseteq \mathbb{P}^3$ $\quad\quad$ $\mathcal{H}_X \subseteq G(1,3)$

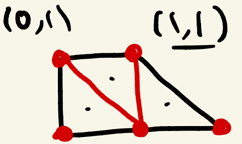
Setup: $X \subseteq \mathbb{P}^n$ codim d. Consider

$$\overline{\{(p,L) : L \text{ tangent to } X \text{ at } p \in X_{reg}\}} \subseteq X \times \mathbb{G}(d,n).$$

The Hurwitz locus $\mathcal{H}_X$ is the projection to $\mathbb{G}(d,n)$.

eg: X is the toric variety given by

$$\varphi : (\mathbb{C}^*)^2 \hookrightarrow \mathbb{P}^4$$
$$(x,y) \longmapsto [1 : x : x^2 : y : xy]$$



deg 3

$\mathbb{G}(2,4)$

$\cup\!|$

$$\mathcal{H}_X = \overline{\{L : L \cap X < 3\}}$$

$$\Updownarrow$$

$$\# V\begin{pmatrix} a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 xy \\ b_0 + \cdots + b_4 xy \end{pmatrix} < 3 \quad \text{where}$$

$$L = \ker \begin{bmatrix} a_0 & \cdots & a_4 \\ b_0 & \cdots & b_4 \end{bmatrix}$$

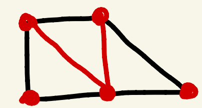
<u>Setup</u>: $X \subseteq \mathbb{P}^n$ codim d. Consider

$$\overline{\{(p,L) : L \text{ tangent to } X \text{ at } p \in X_{reg}\}} \subseteq X \times G(d,n).$$

The <u>Hurwitz locus</u> $\mathcal{H}_X$ is the projection to $G(d,n)$.

<u>eg</u>: X is the toric variety given by

$$\begin{aligned} (\mathbb{C}^*)^2 &\longrightarrow \mathbb{P}^4 \\ (x,y) &\longmapsto [1:x:x^2:y:xy] \end{aligned}$$



Then

- $\mathcal{H}_X = \overline{\left\{L = \ker\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix} : \# L \cap X < 3\right\}} \subseteq G(2,4)$

- $\mathcal{H}_X(a_i, b_i) = 0 \iff$ # sol'ns to $\begin{matrix} a_0 + a_1 x + \dots + a_4 xy \\ b_0 + b_1 x + \dots + b_4 xy \end{matrix} < 3$

Warning

hurwitz Form

- Equations are <u>hard</u>:

p_(0,2,3) ^2*p_(1,2,3) ^2-4*p_(0,1,3) *p_(1,2,3) ^3+p_(0,2,3) ^2*p_(0,2,4) ^2+p_(0,1,4) ^2
*p_(0,2,4) ^2-2*p_(0,1,3)*p_(0,2,4) ^3-2*p_(0,2,3) ^3*p_(1,2,4)+8*p_(0,1,3) *p_(0,2, 3) *p_(1,2,3)*p_(1,2,4) -4*p_(0,1,4) ^3*p_(1,2,4)
-2*p_(0,1,3) *p_(0,2,3) *p_(0,2,4) *p_(1,2,4) +8*p_(0,1,3) *p_(0,1,4) *p_(0,2,4) *p_(1,2,4) -8*p_(0,1,3) ^2*p_(1,2,4) ^2-
2*p_(0,1, 2) *p_(0,2,4) ^2*p_(0,3,4) -2*p_(0,1,2) *p_(0,2,3)*p_(1,2,3) *p_('1,3,4) +8
*p_(0,1,2) *p_(0,2,3) *p_(0,2,4)*p_(1,3,4)+10*p_(0, 1,2)*pL(,1,4)*p_(0,2,4) *p_(1, 3
, 4) - 20*p_(0,1,2)*p_(0,1,3)*p_(1,2,4)*p_(1,3,4) +p_(0,1,2) ^2*p_(1,3,4) ^2-2*p_(0,1, 2) *p_(0,2, 3) ^2*p_(2,3, 4)+12*p_(0,1,2)*p_(0,1,3)
*p_(1,2, 3) *p_(2,3,4) - 12*p_(0,1,2
) *p_(0,1,4) ^2*p_(2,3,4)+12*p_(0,1,2) ^2*p_(0,3,4)*p_(2,3,4)

- Degrees are <u>easier</u>:

$\boxed{\text{Thm}}$ (Sturmfels '17) Let $X \subseteq \mathbb{P}^n$ be irreducible, degree ≥ 2 & codim $\operatorname{Sing}(X) \geq 2$. Let $d = \deg X$, $g =$ <u>sectional</u> genus of X.
Then

$$\deg H_X = 2d + 2g - 2.$$

$g_{geom}(L \cap X)$
$\dim L = d+1$

Any Qs?

Notation:

- For $\vec{n} = (n_1, \dots, n_\ell)$ define multiprojective space

$$\underline{\underline{\mathbb{P}}} := \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_\ell}$$

- For $\vec{\alpha} = (\alpha_1, \dots, \alpha_\ell)$ define

$$\underline{\underline{\mathbb{G}_\alpha}} := \mathbb{G}(\alpha_1, n_1) \times \dots \times \mathbb{G}(\alpha_\ell, n_\ell)$$

- Varieties have multidegree

$$[X] = \sum_{|\alpha| = \mathrm{codim}\, d} \delta_\alpha t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} \quad \in H^*(\mathbb{P})$$

- $\pi_S : \mathbb{P} \dashrightarrow \prod_{i \in S} \mathbb{P}^{n_i}$

Setup:

- $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ codim d
- $\alpha = (\alpha_1, \dots, \alpha_\ell)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_\ell = d$

Hurwitz-Lam forms for $X \subseteq G(k,n)$

The multigraded Hurwitz incidence is

$$\Phi_X^\alpha = \overline{\left\{ (p,L) : \begin{array}{l} L = L_1 \times \cdots \times L_\ell \text{ tangent to } X \text{ at } p \in X_{reg} \\ \text{but } \pi_S(L) \text{ not tangent to } \pi_S(X) \end{array} \right\}} \subseteq \mathbb{P} \times G_\alpha .$$

The multigraded Hurwitz locus H_X^α is the projection to G_α.

eg: X = toric four-fold $\qquad \alpha = (2,0), (0,2), \underline{\underline{(1,1)}}$

$$(\mathbb{C}^*)^4 \longrightarrow \mathbb{P}^3 \times \mathbb{P}^3$$

$$(x,y,z,w) \longmapsto [xy:z:w:1] \times [zw:x:y:1]$$

- Write $L_1 = \ker \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$, $L_2 = \ker \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{bmatrix}$. $\quad L = L_1 \times L_2$

deg $(6, *)$

- Then $H_X^{(1,1)}(a_i, b_i, c_i, d_i) = 0$ in $G(1,3) \times G(1,3)$

$$\Longleftrightarrow \begin{cases} a_0 xy + a_1 z + a_2 w + a_3 = b_0 xy + b_1 z + b_2 w + b_3 = 0 \\ c_0 zw + c_1 x + c_2 y + c_3 = d_0 zw + d_1 x + d_2 y + d_3 = 0 \end{cases}$$ has < 4 sol'ns

$H_X^{(1,1)}$ is a mixed discriminant [Cattani-Cueto-Dickenstein-Di Rocco-Sturmfels '11]

Degrees & Genera

The multidegree of $X \subseteq \mathbb{P}$ codim d is

$$[X] = \sum_{|\alpha|=d} \delta_\alpha T_1^{\alpha_1} \cdots T_\ell^{\alpha_\ell}$$

The multisectional genus for $|\beta| = d+1$ is

$$g_\beta := g_{geom}(X \cap L_1 \times \cdots \times L_\ell) \qquad \dim L_i = \beta_i$$

L_i generic.

$\boxed{\text{Thm}}$ (P-Sodomaco-Sturmfels '26)

Suppose $\delta_\alpha \geq 2$ and $n_i \geq \max(2, 1+\alpha_i)$.

The multigraded Hurwitz form H_X^α is an irred variety in G_α of degree $(u_1, \ldots, u_\ell)$, where

$$u_i \leq 2(g_{\alpha+e_i} + \delta_\alpha - 1).$$

{Any Qs?

Def'n: A curve $C \subseteq \mathbb{P}$ is polynodal if $\pi_S(C)$ has at most nodal singularities for any $S \subseteq [\ell]$.

$\subseteq \mathbb{P}^2 \times \mathbb{P}^2$ $\subseteq \mathbb{P}^2$

not allowed

A variety $X \subseteq \mathbb{P}$ is polynodal if its general linear sections are.

$\boxed{\text{Thm}}$ (P-Sodomaco-Sturmfels '26)

Suppose $\delta_\alpha \geq 2$ and $n_i \geq \max(2, 1+\alpha_i)$ and X is polynodal.

Then

$$u_i = 2(g_{\alpha+e_i} + \delta_\alpha - 1).$$

$u_i = 6 < 8$

(non-eg: toric 4-fold)

Toric varieties

- Suppose $P_1, \dots, P_\ell$ are polytopes in $\mathbb{Z}^d$, for $d \geq \ell$

  $\Rightarrow X \subseteq \mathbb{P}$ with $\underline{\underline{n_i}} = |\underline{\underline{P_i \cap \mathbb{Z}^d}}|$

- write

  minkowski sum

$$\mathrm{Vol}(T_1 P_1 + \dots + T_\ell P_\ell) =: \sum_{|\gamma| = d} \underline{\underline{\mu_\gamma}} \, T_1^{\gamma_1} \cdots T_\ell^{\gamma_\ell}$$

mixed volumes of $P_1, \dots, P_\ell$

Volume polynomial

$\boxed{\text{Thm}}$ (Bernstein)

$$[X] = \sum \mu_\gamma (\gamma_1! \cdots \gamma_\ell!) \, T_1^{n - \gamma_1} \cdots T_\ell^{n - \gamma_\ell}$$

$\boxed{\text{Thm}}$ (Khovanskii)

$$g_\beta(X) = \sum_{0 \leq \gamma \leq \beta} (-1)^{|\beta - \gamma|} \, |\mathrm{int}(\gamma_1 P_1 + \dots + \gamma_\ell P_\ell) \cap \mathbb{Z}^d|$$

Rmk: Nash discriminants are of this form.

open Q: when is X polynodal?

Discriminants in Physics

- In dual momentum space, particles are represented by points in $G(1,3)$ $\supseteq \mathbb{R}^{1,3}$

- Momentum vectors $L_i = \text{rowspan}\, \hat{L}_i$ have an inner product:

  ↳ 2×4 matrix

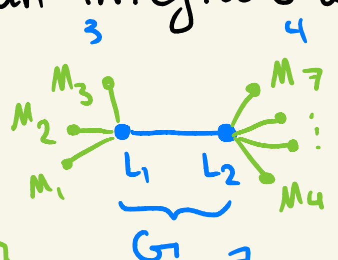
$$\langle L_1, L_2 \rangle := \det_4 \begin{bmatrix} - \hat{L}_1 - \\ \\ - \underset{4}{\hat{L}_2} - \end{bmatrix} = P_{12}(L_1)P_{34}(L_2) - P_{13}(L_1)P_{24}(L_2) + \cdots$$

↳ extends to $\mathbb{P}^5$

where $\langle L_1, L_2 \rangle = 0 \iff L_1$ meets L_2 in $\mathbb{P}^3$

- Feynman integrals are associated to (dual Feynman) graphs:

at most 5 at each vertex



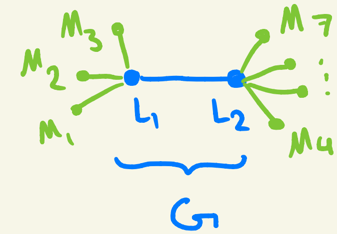
$$\rightsquigarrow \int_\Gamma \frac{N(M_i, L_j)}{\underbrace{\langle M_1, L_1 \rangle \cdots \langle M_7, L_2 \rangle}_{\text{parameters}} \langle L_1, L_2 \rangle} \, dL_1 \, dL_2$$

$4V - E$ $=7$ pendant edges, $5 - \alpha_i$ at v_i

Hurwitz meets Feynman

Q: For which M_i does

$$I_{G,\alpha} := \int \frac{N(M_i, L_j)}{\langle M_1,L_1\rangle \cdots \langle M_7,L_2\rangle\langle L_1,L_2\rangle}\, dL_1\, dL_2$$



develop poles & branch cuts?

- $V_G := \{(L_1, \dots, L_\ell) : \text{if } (ij) \in G, \text{ then } \langle L_i, L_j\rangle = 0\}$. $\subseteq G(1,3)^\ell \subseteq (\mathbb{P}^5)^\ell$

  $G(1,3) \hookrightarrow \mathbb{P}^5$

- codim $V_G = \ell + |E_G|$ in $(\mathbb{P}^5)^\ell$

  $= 2 + 1 = 3$

The Hurwitz form $H^\alpha_{V_G}$ is the leading singularity discriminant of the Feynman integral $I_{G,\alpha}$

physicists

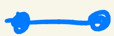
[Hollering–Mazzucchelli–Parisi–Sturmfels '26]

$\boxed{\text{Thm}}$ (P-Sodomaco-Sturmfels)

Write $\sum \delta_\alpha T^\alpha := 2^\ell T_1 \cdots T_\ell \prod_{ij \in G} (T_i + T_j)$

The degree of $H^\alpha_{V_G}$ is bounded by

$$u_i \leq 2\delta_\alpha + \sum_{\substack{j \in [\ell] \\ \alpha_j + \hat{\delta}_{ij} > 0}} \left(\delta_{\alpha + e_i - e_j} - \hat{\delta}_{ij} + \mathrm{degree}_G(i) \right)$$

<u>eg</u>: 
G , $\alpha = (2,1)$

Then $H^{(2,1)}_{V_G} \subseteq G(1,5) \times G(2,5)$ degree $(4,4)$

But $\delta_\alpha = 4 \Rightarrow$ bounded by $(4,8)$.

Thanks for listening!